REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 28, N° 2, 1999, pp. 117–123

ABOUT SOME INTERPOLATION FORMULAS OVER TRIANGLES

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1. PRELIMINARIES

Beginning with the paper by Barnhill, Birkhoff and Gordon [1], the interpolation problem to boundary data on a triangle was largely studied. Considering the standard triangle $T_h = \{(x,y)\in R^2 \,|\, x \geq 0,\, y \geq 0,\, x+y \leq h\}$ with the vertices $V_1 = (h, 0)$, $V_2 = (0, h)$, $V_3 = (0, 0)$ and the opposite sides denoted by E_1, E_2, E_3 (fig. 1), in [1] there are constructed some interpolants which match a given function $f : T_h \to R$ on the sides of the triangle T_h.

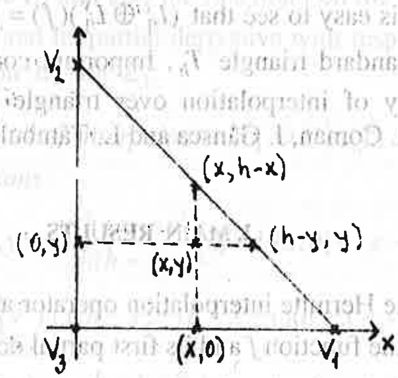

Fig. 1

The ideas from [1] will be shortly presented. Let be $(x, y) \in int\,T_h$ and let be L_1^x, L_1^y the linear operators along the parallel to the side E_2, respectively to the side E_1, i.e.

$$L_1^x(f)(x,y) = Ax + B \qquad L_1^y(f)(x,y) = Cy + D, \tag{1.1}$$

where the coefficients A, B, C, D are determined from the conditions

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1991 AMS Subject Classification: 41-00, 41A10, 41A25.
Key words: blending interpolant, boolean sum operator.

$$L_1^x(f)(0,y)=f(0,y),\quad L_1^x(f)(h-y,y)=f(h-y,y) \tag{1.2.1}$$

$$L_1^y(f)(x,0)=f(x,0),\quad L_1^y(f)(x,h-x)=f(x,h-x) \tag{1.2.2}$$

Solving the linear algebraic systems (1.2.1), (1.2.2) the operators become:

$$L_1^x(f)(x,y)=\frac{h-x-y}{h-y}\cdot f(0,y)+\frac{x}{h-y}\cdot f(h-y,y) \tag{1.3.1}$$

$$L_1^y(f)(x,y)=\frac{h-x-y}{h-x}\cdot f(x,0)+\frac{y}{h-x}\cdot f(x,h-x) \tag{1.3.2}$$

By direct substitution the equalities (1.2.1), (1.2.2) follow for any $x, y \in [0, h)$, which prove that $L_1^x(f)$ interpolates the function f on the sides E_1, E_3 of T_h and $L_1^y(f)$ interpolates the function f on the sides E_1 and E_3 of T_h.

The boolean sum operator $L_1^x\oplus L_1^y=L_1^x+L_1^y-L_1^xL_1^y$ (see [8]) defines the so called "blending interpolant" $(L_1^x\oplus L_1^y)(f)$. By simply computation

$$(L_1^x\oplus L_1^y)(f)(x,y)=\frac{h-x-y}{h-y}\cdot f(0,y)+\frac{h-x-y}{h-x}\cdot f(x,0)+\frac{y}{h-x}\cdot f(x,h-x)-\frac{h-x-y}{h}\cdot f(0,0)-\frac{y(h-x-y)}{h(h-y)}\cdot f(0,h) \tag{1.4}$$

From (1.4) it is easy to see that $(L_1^x\oplus L_1^y)(f)=f$ on ∂T_h where ∂T_h is the boundary of the standard triangle T_h. Important contributions to the development of the theory of interpolation over triangle are due to the Romanian mathematicians Gh. Coman, I. Gânsca and L. Tâmbulea [5], [6], [7], [8].

2. MAIN RESULTS

Let H_2^y be the Hermite interpolation operator along the parallel to the side E_1 which matches the function f and its first partial derivative with respect to y at the point $(x, 0)$ and also the function f at $(x, h-x)$. Let H_2^x be the Hermite interpolation operator along the parallel to the side E_2 which matches the function f and its first partial derivative with respect to x at the point $(0, y)$ and also the function f at $(h-y, y)$.

LEMMA 2.1. *The equalities*

$$H_2^y(f)(x,y)=\frac{y(h-x-y)}{h-x}\cdot f^{(0,1)}(x,0)+\frac{y^2}{(h-x)^2}\cdot f(x,h-x)+\frac{(h-x-y)(h-x+y)}{(h-x)^2}\cdot f(x,0) \tag{2.2.1}$$

$$H_2^x(f)(x,y)=\frac{x(h-x-y)}{h-y}\cdot f^{(1,0)}(0,y)+\frac{x^2}{(h-y)^2}\cdot f(h-y,y)+\frac{(h-x-y)(h-x+y)}{(h-y)^2}\cdot f(0,y) \tag{2.2.2}$$

hold for any $x,y\in[0,h)$.

Proof. Clearly $H_2^y(f)$ is a polynomial of degree two with respect to y and $H_2^x(f)$ is a polynomial of degree two with respect to x, i.e. $H_2^y(f)(x,y)=Ay^2+By+C$ and $H_2^x(f)(x,y)=Dx^2+Ex+F$ which satisfy the interpolation conditions:

$$H_2^y(f)(x,0)=f(x,0),\ H_2^y(f)^{(0,1)}(x,0)=f^{(0,1)}(x,0),\ H_2^y(f)(x,h-x)=f(x,h-x). \tag{2.3.1}$$

$$H_2^x(f)(0,y)=f(0,y),\ H_2^x(f)^{(0,1)}(0,y)=f^{(0,1)}(0,y),\ H_2^x(f)(h-y,y)=f(h-y,y). \tag{2.3.2}$$

Using these conditions one arrives to (2.2.1) and (2.2.2).

Remark 2.2. $H_2^y(f)$ interpolates the functions f and its partial derivative with respect to y on the side E_1 and also the function f on the side E_3 and $H_2^x(f)$ interpolates the function f and its partial derivative with respect to x on the side E_2 and also the function f on the side E_3.

LEMMA 2.3. *The blending interpolants* $(L_1^x\oplus H_2^y)(f)$ *and* $(L_1^y\oplus H_2^x)(f)$ *have the following expressions:*

$$\begin{aligned}(L_1^x\oplus H_2^y)(f)(x,y)=&\frac{y(h-x-y)}{h(h-x)}\{hf^{(0,1)}(x,0)+(x-h)f^{(0,1)}(0,0)\}+\\&+\frac{1}{(h-x)^2}\{y^2f(x,h-x)+(h-x-y)(h-x+y)f(x,0)\}+\\&+\frac{h-x-y}{h-y}f(0,y)-\frac{h-x-y}{h^2(h-y)}\{y^2f(0,h)+(h^2-y^2)f(0,0)\},\end{aligned} \tag{2.4.1}$$

$$\begin{aligned}(L_1^y\oplus H_2^x)(f)(x,y)=&\frac{x(h-x-y)}{h(h-y)}\{hf^{(1,0)}(0,y)+(y-h)f^{(1,0)}(0,0)\}+\\&+\frac{1}{(h-y)^2}\{x^2f(h-y,y)+(h-x-y)(h-x+y)f(0,y)\}+\\&+\frac{h-x-y}{h-x}f(x,0)-\frac{h-x-y}{h^2(h-x)}\{x^2f(h,0)+(h^2-x^2)f(0,0)\}.\end{aligned} \tag{2.4.2}$$

Proof. First one calculates the product $(L_1^xH_2^y)(f)$ and one obtains:

$$(L_1^x H_2^y)(f)(x,y) = \frac{h-x-y}{h-y} H_2^y(f)(0,y) + \frac{x}{h-y} H_2^y(f)(h-y,y) =$$

$$= \frac{y(h-x-y)}{h} f^{(0,1)}(0,0) + \frac{y^2(h-x-y)}{h^2(h-y)} f(0,h) +$$

$$+ \frac{(h-x-y)(h+y)}{h^2} f(0,0) + \frac{x}{h-y} f(h-y,y)$$

Then, taking into account that $L_1^x \oplus L_1^y = L_1^x + L_1^y - L_1^x L_1^y$ it follows that (2.4.1) holds. In a similar way, (2.4.2) holds.

THEOREM 2.1. *The operators* $L_1^x \oplus H_2^y$ *and* $L_1^y \oplus H_2^x$

(2.5.1) $(L_1^x \oplus H_2^y)(f) = f$ *on* ∂T_h, $(L_1^x \oplus H_2^y)^{(0,1)}(f) = f^{(0,1)}$ *on* E_1,

(2.5.2) $(L_1^y \oplus H_2^x)(f) = f$ *on* ∂T_h, $(L_1^y \oplus H_2^x)^{(1,0)}(f) = f^{(1,0)}$ *on* E_2.

Proof. By direct substitution one obtains

$$(L_1^x \oplus H_2^y)(f)(x,0) = \frac{1}{(h-x)^2}(h-x)^2 f(x,0) + \frac{h-x}{x} f(0,0) -$$

$$- \frac{h-x}{h^3} h^2 f(0,0) = f(x,0)$$

which proves that $(L_1^x \oplus H_2^y)(f) = f$ on E_2. In a similar way one proves that $(L_1^x \oplus H_2^y)(f) = f$ on E_1 and on E_3. Next, one compute the first partial derivative with respect to y at $(x, 0)$ and one obtains:

$$(L_1^x \oplus H_2^y)^{(0,1)}(f)(x,y) = \frac{h-x-2y}{h(h-x)}\{hf^{(0,1)}(x,0) + (x-h)f^{(0,1)}(0,0)\} +$$

$$\frac{2y}{(h-x)^2}\{f(x,h-x) - f(x,0)\} + \frac{x}{(h-y)^2} f^{(0,1)}(0,y) +$$

$$+ \frac{x}{h^2(h-y)^2}\{y^2 f(0,h) + (h^2-y^2) f(0,0)\} -$$

$$- \frac{h-x-y}{h^2(h-y)}\{2yf(0,h) - 2y \cdot f(0,0)\}$$

So, with $y = 0$, $(L_1^x \oplus H_2^y)^{(0,1)}(f)(x,0) = f^{(0,1)}(x,0) + \frac{x-h}{h} f^{(0,1)}(0,0) -$

$$- \frac{x}{h^2} f(0,0) + \frac{h-x}{h} f^{(0,1)}(0,0) + \frac{x}{h^2} f(0,0) = f^{(0,1)}(x,0)$$

In a similar way, one obtains (2.5.2.).

Remark 2.4. The multiplicity of the knots $(x, 0)$, $(x, h-x)$ and respectively $(0, y)$, $(h-y, y)$ can be inverted.

Remark 2.5. Considering the boolean sum operator $(H_2^x \oplus H_2^y)$ one obtains the blending function interpolant $(H_2^x \oplus H_2^y)(f)$. This function interpolates f on ∂T_h and its partial derivatives $f^{(0,1)}$ and $f^{(1,0)}$ on E_1 and respectively on E_2.

Remark 2.6. The interpolation procedures which were presented above have many applications in computer aided geometry (see [7], [8]).

Using the boolean sum operators we can consider now the following approximation formulas:

$$f = (L_1^x \oplus H_2^y)(f) + R_{12}^{xy} f, \quad f = (L_1^y \oplus H_2^x)(f) + R_{12}^{yx} f,$$

where R_{12}^{xy} and R_{12}^{yx} are the corresponding remainder terms.

THEOREM 2.2. *If* $f \in B_{12}(0, 0)$ *then the remainder term* R_{12}^{xy} *is:*

$$R_{12}^{xy}(f)(x,y) = \int_0^h K_{30}(x,y,s) f^{(3,0)}(s,0)\mathrm{d}s + \int_0^h K_{21}(x,y,s) f^{(2,1)}(s,0)\mathrm{d}s +$$

$$+ \int_0^h K_{03}(x,y,t) f^{(0,3)}(0,t)\mathrm{d}t + \iint_{T_h} K_{12}(x,y,s,t) f^{(1,2)}(s,t)\mathrm{d}s\mathrm{d}t$$

where

$$K_{30}(x,y,s) = \frac{(x-s)_+^2}{2}, \quad K_{21}(x,y,s) = y \cdot (x-s)_+,$$

$$K_{03}(x,y,t) = \frac{y^2}{(h-x)^2} \cdot \frac{(h-x-t)_+^2}{2} + \frac{h-x-y}{h-y} \cdot \frac{(y-t)_+^2}{2} - \frac{y^2(h-x-y)(h-t)^2}{2h^2(h-y)}$$

$$K_{12}(x,y,s,t) = \frac{y}{h-x}(x-s)_+^0 \left[h-x-y + \frac{y}{h-x} \cdot (h-x-t)_+\right].$$

Proof. Taking into account that $R_{12}^{xy} f = f$, $(\forall) f \in P_2^2$, the proof follows by the Sard kernel theorem in triangles [1], with

$$K_{30}(x,y,s) = (L_1^x \oplus H_2^y)\left[\frac{(x-s)_+^2}{2}\right], \quad K_{21}(x,y,s) = (L_1^x \oplus H_2^y)[(x-s)_+ \cdot y],$$

$$K_{03}(x,y,t) = (L_1^x \oplus H_2^y)\left[\frac{(y-t)_+^2}{2}\right],$$

$$K_{12}(x,y,s,t) = (L_1^x \oplus H_2^y)[(x-s)_+^0 \cdot (y-t)_+].$$

THEOREM 2.3. *If $f \in B_{21}(0,0)$ then the remainder term R_{12}^{xy} has the expression*:

$$R_{12}^{xy}(f)(x,y)=\int_0^h K_{30}(x,y,s)f^{(3,0)}(s,0)\mathrm{d}s+\int_0^h K_{12}(x,y,s)f^{(1,2)}(s,0)\mathrm{d}s+$$

$$+\int_0^h K_{03}(x,y,t)f^{(0,3)}(0,t)\mathrm{d}t+\iint_{T_h} K_{21}(x,y,s,t)f^{(2,1)}(s,t)\mathrm{d}s\mathrm{d}t$$

where

$$K_{30}(x,y,s)=\frac{(y-s)_+^2}{2},\quad K_{12}(x,y,s)=x\cdot(y-s)_+,$$

$$K_{03}(x,y,t)=\frac{x^2}{(h-y)^2}\cdot\frac{(h-y-s)_+^2}{2}+\frac{h-x-y}{h-x}\cdot\frac{(x-s)_+^2}{2}-\frac{x^2(h-x-y)(h-s)^2}{2h^2(h-x)}$$

$$K_{12}(x,y,s,t)=\frac{x}{h-y}(y-t)_+^0\left[h-x-y+\frac{x}{h-y}\cdot(h-y-s)_+\right].$$

Proof. Taking into account that $R_{12}^{yx}f=f$, $(\forall)\, f\in P_2^2$, the proof follows by the Sard kernel theorem in triangles [1], with

$$K_{30}(x,y,s)=(L_1^x\oplus H_2^y)\left[\frac{(x-s)_+^2}{2}\right],\quad K_{12}(x,y,t)=(L_1^x\oplus H_2^y)[(y-t)_+\cdot x],$$

$$K_{03}(x,y,t)=(L_1^y\oplus H_2^x)\left[\frac{(y-t)_+^2}{2}\right],$$

$$K_{21}(x,y,s,t)=(L_1^y\oplus H_2^y)[(x-s)_+\cdot(y-t)_+^0].$$

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Received December 10, 1997

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