# ON SOME INTERPOLATORY ITERATIVE METHODS FOR THE SECOND DEGREE POLYNOMIAL OPERATORS (II)

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## 1. THE APPROXIMATION OF THE SOLUTIONS OF RICCATI DIFFERENTIAL EQUATIONS

Consider the differential equation

(1.1) $$y' = a_0(x)y^2 + a_1(x)y + a_2(x),$$

where the applications $a_0, a_1, a_2 : [a,b] \to \mathbb{R}$ are continuous on the interval $[a,b]$, $a < b$ and $a_2$ is not the null function.

We are interested to approximate the solution of (1.1) with the condition

(1.2) $$y(a) = y_a.$$

The solution will be searched in the space $C^1[a,b]$. For this purpose we consider the operator $F : C^1[a,b] \to C[a,b]$ given by

(1.3) $$F(y)(x) = y'(x) - a_0(x)y^2(x) - a_1(x)y(x) - a_2(x).$$

The first and second order divided differences of $F$ are given by

(1.4) $$[y_1, y_2; F]h(x) = h'(x) - \big(a_0(x)(y_1(x) + y_2(x)) + a_1(x)\big)h(x),$$

respectively

(1.5) $$[y_1, y_2, y_3; F]h(x)k(x) = -a_0(x)h(x)k(x),$$

for all $y_1, y_2, y_3, h, k \in C^1[a,b]$.

Formula (1.5) shows that the divided differences of order higher than two are the null multilinear operators.

Let $y_0, y_1 \in C^1[a,b]$ be given. The chord iterates $y_k \in C^1[a,b]$, $k = 2, 3, \ldots$ are constructed by

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$$(1.6) \qquad [y_{k-1}, y_k; F](y_{k+1} - y_k) + F(y_k) = 0, \; k = 1, 2, \ldots$$

As we shall see, the above equation lead to some first order linear nonhomogeneous differential equations.

Replacing in (1.6) the expression (1.4) of $F$ we get for $y_{k+1}$ the following differential equation.

$$(1.7) \qquad y'_{k+1}(x) = \varphi_k(x) y_{k+1}(x) + \psi_k(x),$$

where we have denoted

$$\varphi_k(x) = a_0(x)(y_{k-1}(x) + y_k(x)) + a_1(x) \text{ and }$$

$$\psi_k(x) = -a_0(x) y_k(x) y_{k-1}(x) + a_2(x).$$

For solving (1.7) consider the condition

$$(1.8) \qquad y_{k+1}(a) = y_a,$$

so we are led to the expression of $y_{k+1}$:

$$(1.9) \qquad y_{k+1}(x) = \exp\left(\int_a^x \varphi_k(s) ds\right)\left(y_a + \int_a^x \psi_k(s) \exp\left(-\int_a^s \varphi_k(t) dt\right) ds\right).$$

We are interested to final a bound for $\|[y_0, y_1; F]^{-1}\|$.

For this purpose consider the problem

$$(1.2) \qquad \begin{cases} [y_0, y_1; F] h(x) = u(x), \; u \in C[a, b] \\ h(a) = 0 \end{cases}$$

which has the solution

$$(1.3) \qquad h(x) = \exp\left(\int_a^x \varphi_1(s) ds\right) \int_a^x u(s) \exp\left(-\int_a^s \varphi_1(t) dt\right) ds.$$

This equality yields

$$(1.10) \qquad \|[y_0, y_1; F]^{-1}\| \le \sup_{x \in [a,b]} \exp\left(\int_a^x \varphi_1(s) ds\right) \int_a^x \exp\left(-\int_a^s \varphi_1(t) dt\right) ds.$$

For the second order divided differences we have

$$(1.11) \qquad \|[u, v, w; F]\| \le \sup_{x \in [a,b]} |a_0(x)|.$$

Now denote

$$\bar{b}_0 = \sup_{x \in [a,b]} \exp\left(\int_a^x \varphi_1(s) ds\right) \int_a^x \exp\left(-\int_a^s \varphi_1(t) dt\right) ds;$$

$$\bar{\alpha} = \sup_{x \in [a,b]} |a_0(x)|$$

$$\bar{d}_0 = \sup_{x \in [a,b]} |y_1(x) - y_0(x)|$$

$$\bar{B}_{\bar{r}}(y_0) = \left\{ y \in C^1[a,b] \; \Big| \; \sup_{x \in [a,b]} |y(x) - y_0(x)| \le \bar{r} \right\}.$$

With the above notations and taking into account relations (1.10) and (1.11), Theorem 4.1 from [5] implies:

THEOREM 1.1. *Assume the functions* $y_0, y_1 \in C^1[a, b]$, $a_0, a_1, a_2 \in C[a, b]$ *and the real positive numbers* $\bar{b}_0, \bar{\alpha}, \bar{d}_0$ *and* $\bar{r}$ *satisfy the following conditions:*

*a)* $\bar{b}_0 \bar{\alpha}(2\bar{r} + \bar{d}_0) = q < 1;$

*b)* $\varepsilon_0 = \bar{\alpha} \bar{b} \; \sup\limits_{x \in [a,b]} |F(y_0)(x)| < 1$, $\bar{\alpha} \bar{b} \sup\limits_{x \in [a,b]} |F(y_1)(x)| < \varepsilon_0^l$, *where* $\bar{b} = \dfrac{\bar{b}_0}{1 - q}$ *and*

$$l = \frac{1 + \sqrt{5}}{2};$$

*c)* $\dfrac{\varepsilon_0^l}{\bar{\alpha} \bar{b}(1 - \varepsilon_0)} + \bar{d}_0 \le \bar{r}.$

*Then the sequence* $(y_k)_{k \ge 0}$ *given by (1.9) converges uniformly and its elements lie in the ball* $\bar{B}_{\bar{r}}(y_0)$. *Denoting* $y^* = \lim\limits_{k \to \infty} y_k$, *then* $y^*$ *is a solution of (1.1)–(1.2). Moreover, the following estimation hold:*

$$\sup_{x \in [a,b]} |y^*(x) - y_k(x)| \le \frac{\varepsilon_0^{l^k}}{\bar{\alpha} \bar{b}(1 - \varepsilon_0^l)}, \quad k = 2, 3, \ldots$$

## 2. THE APPROXIMATION OF THE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

Consider the integral equation

$$(2.1) \qquad \varphi(t) = \lambda \int_a^b a_1(t, s) \varphi(s) ds + \mu \int_a^b a_2(t, s) \varphi^2(s) ds + f(t),$$

where $\lambda, \mu \in \mathbb{R}$, $a_1, a_2 \in C[a, b]^2$ and $f \in C[a, b]$.

Here we take the operator $F : C[a, b] \to C[a, b]$ given by

$$(2.2) \qquad F(\varphi)(t) = \varphi(t) - \lambda \int_a^b a_1(t, s) \varphi(s) ds - \mu \int_a^b a_2(t, s) \varphi^2(s) ds - f(t).$$

The first and the second order divided differences of $F$ on $u, v, w \in C[a, b]$ are given by the following relations:

$$(2.3) \qquad [u, v; F]h(t) = h(t) - \lambda \int_a^b a_1(t,s)h(s)\,ds - \mu \int_a^b a_2(t,s)(u(s) + v(s))h(s)\,ds$$

$$[u, v, w; F]h(t)k(t) = -\mu \int_a^b a_2(t,s)h(s)k(s)\,ds$$

where $h, k \in C[a, b]$.

Let $\varphi_0, \varphi_1 \in C[a, b]$ be given. As in the previous section, we shall construct the sequence $(\varphi_k)_{k \geq 0} \subset C[a,b]$ with the solutions of the following linear integral equations

$$(2.4) \qquad [\varphi_{k-1}, \varphi_k; F](\varphi_{k+1} - \varphi_k) + F(\varphi_k) = 0, \quad k = 1, 2, \dots$$

From (2.4), by (2.2) and (2.3) we get for $\varphi_{k+1}$ the following linear integral Fredholm equation

$$(2.5) \qquad \varphi_{k+1}(t) = \lambda \int_a^b a_1(t,s)\varphi_{k+1}(s)\,ds + \\ + \mu \int_a^b a_2(t,s)(\varphi_{k-1}(s) + \varphi_k(s))\varphi_{k+1}(s)\,ds + G_k(t)$$

where

$$(2.6) \qquad G_k(t) = f(t) - \mu \int_a^b a_2(t,s)(\varphi_{k-1}(s) + \varphi_k(s))\,ds .$$

In order to consider Theorem 4.1 from [5] for the convergence of the sequence $(\varphi_k)_{k \geq 0}$ we need a bound for the operator $[\varphi_0, \varphi_1; F]^{-1}$. In this respect we take the following equation

$$[\varphi_0, \varphi_1; F]h(t) = u(t), \quad \text{with } \lambda \neq 0,$$

i.e.

$$(2.7) \qquad h(t) = \lambda \int_a^b a_1(t,s)h(s)\,ds + \mu \int_a^b a_2(t,s)(\varphi_0(s) + \varphi_1(s))h(s)\,ds + u(t).$$

We shall assume that this equation has a unique solution. Denote $K_0\left(t, s, \lambda, \frac{\mu}{\lambda}\right)$ the corresponding "resolvent" kernel. Then the solution of (2.7) has the following form:

$$(2.8) \qquad h(t) = u(t) + \lambda \int_a^b K_0\left(t, s, \lambda, \frac{\mu}{\lambda}\right)u(s)\,ds.$$

It can be easily seen by the above considerations that the following relation hold:

$$(2.9) \qquad \left\| [\varphi_0, \varphi_1; F]^{-1} \right\| \leq 1 + |\lambda|(b-a) \sup_{a \leq t, s \leq b} \left| K_0\left(t, s, \lambda, \frac{\mu}{\lambda}\right) \right|.$$

At the same time, the norm of $[u, v, w; F]$ is bounded by

$$\left\| [u, v, w; F] \right\| \leq |\mu|(b-a) \sup_{a \leq t, s \leq b} |a_2(t,s)|.$$

We make the following notations:

$$\tilde{b}_0 = 1 + |\lambda|(b-a) \sup_{a \leq t, s \leq b} \left| K_0\left(t, s, \lambda, \frac{\mu}{\lambda}\right) \right|$$

$$\tilde{\alpha} = |\mu|(b-a) \sup_{a \leq t, s \leq b} |a_2(t,s)|$$

$$\tilde{d}_0 = \sup_{t \in [a,b]} |\varphi_1(t) - \varphi_0(t)|$$

$$\bar{B}_{\tilde{r}}(\varphi_0) = \left\{ \varphi \in C[a,b] \mid \sup_{t \in [a,b]} |\varphi(t) - \varphi_0(t)| \leq \tilde{r} \right\}.$$

Theorem 4.1 from [5] then implies the next result.

THEOREM 2.1 *Assume the functions* $a_1, a_2 \in C[a, b]^2$ *and the real numbers* $\lambda, \mu, \tilde{b}_0, \tilde{\alpha}, \tilde{d}_0$ *and* $\tilde{r}$ *satisfy the following conditions:*

*a)* $\tilde{b}_0 \tilde{\alpha}(2\tilde{r} + \tilde{d}_0) = \tilde{q} < 1$;

*b)* $\varepsilon_0 = \tilde{\alpha}\,\tilde{b}^2 \|F(\varphi_0)\| < 1$, $\rho_1 = \tilde{\alpha}\,\tilde{b}\|F(\varphi_1)\| \leq \varepsilon_0^l$, *where*

$$\|F(\varphi_i)\| = \sup_{t \in [a,b]} |F(\varphi_i)(t)|, \quad i = 1, 2; \quad \tilde{b} = \frac{\tilde{b}_0}{1 - \tilde{q}} \text{ and } l = \frac{1 + \sqrt{5}}{2};$$

*c)* $\dfrac{\varepsilon_0^l}{\tilde{\alpha}\,\tilde{b}(1 - \varepsilon_0)} + \tilde{d}_0 \leq \tilde{r}.$

*Then the sequence* $(\varphi_k)_{k \geq 0}$ *generated by (2.5) converges uniformly and its elements lie in* $\bar{B}_{\tilde{r}}(\varphi_0)$. *Denoting* $\varphi^* = \lim_{k \to \infty} \varphi_k$, *then* $\varphi^*$ *is a solution of the integral equation (2.1) and the following estimations hold:*

$$\sup_{t \in [a,b]} |\varphi^*(t) - \varphi_k(t)| \leq \frac{\varepsilon_0^{l^k}}{\tilde{\alpha}\,\tilde{b}(1 - \varepsilon_0^{l^k})}, \quad k = 2, 3, \dots$$

As we have seen, the chord method requires the solving of a linear integral Fredholm equation at each iteration step. The problem takes a simplified form from the practical viewpoint when the kernels $a_1$ and $a_2$ from (2.1) are degenerate. In this case the linear integral equations (2.5) will be also with degenerate kernels, as can be easily seen. We shall consider in the following this particular case.

Let $\alpha_i, \beta_i \in C[a, b], i = 1, \ldots, p$ two sets containing $p$ linear independent functions and also $\gamma_i, \delta_i \in C[a, b], i = 1, \ldots, m$ two other sets with the same properties.

Assume the kernels $a_1$ and $a_2$ have the following form:

$$a_1(t, s) = \sum_{i=1}^{p} \alpha_i(t) \beta_i(s) \quad \text{and}$$

$$a_2(t, s) = \sum_{i=1}^{m} \gamma_i(t) \delta_i(s).$$

It can be verified without difficulty that the solution $\varphi_{k+1}$ of equation (2.5) has the form

$$(2.10) \quad \begin{aligned} \varphi_{k+1}(t) = {} & \lambda \sum_{i=1}^{p} \alpha_i(t) \int_{b}^{a} \beta_i(s) \varphi_{k+1}(s) \, \mathrm{d}s + \\ & + \mu \sum_{i=1}^{m} \gamma_i(t) \int_{b}^{a} \delta_i(s) \left( \varphi_{k-1}(s) + \varphi_k(s) \right) \varphi_{k+1}(s) \, \mathrm{d}s + G_k(t). \end{aligned}$$

Consider the following notations:

$$x_i^{(k+1)} = \int_a^b \beta_i(s) \varphi_{k+1}(s) \, \mathrm{d}s, \qquad\qquad i = 1, \ldots, p$$

$$y_i^{(k+1)} = \int_a^b \delta_i(s) \left( \varphi_{k-1}(s) - \varphi_k(s) \right) \varphi_{k+1}(s) \, \mathrm{d}s, \quad i = 1, \ldots, m$$

$$a_{ij} = \int_a^b \alpha_i(t) \beta_j(t) \, \mathrm{d}t, \qquad\qquad i, j = 1, \ldots, p$$

$$b_{ij} = \int_a^b \gamma_i(t) \beta_j(t) \, \mathrm{d}t, \qquad\qquad i = 1, \ldots, m, \; j = 1, \ldots, p$$

$$A_{ji}^{(k+1)} = \int_a^b \alpha_i(t) \delta_j(t) \left( \varphi_k(t) + \varphi_{k-1}(t) \right) \mathrm{d}t, \quad i = 1, \ldots, p, \; j = 1, \ldots, m$$

$$B_{ji}^{(k+1)} = \int_a^b \gamma_i(t) \delta_j(t) \left( \varphi_k(t) + \varphi_{k-1}(t) \right) \mathrm{d}t, \quad i, j = 1, \ldots, m$$

$$\theta_j^{(k+1)} = \int_a^b \beta_j(t) G_k(t) \, \mathrm{d}t, \qquad\qquad j = 1, \ldots, p$$

$$\varepsilon_j^{(k+1)} = \int_a^b \delta_j(t) \left( \varphi_k(t) + \varphi_{k-1}(t) \right) G_k(t) \, \mathrm{d}t, \quad j = 1, \ldots, m$$

We obtain the for $\varphi_{k+1}$ the expression

$$(2.11) \quad \varphi_{k+1}(t) = \lambda \sum_{i=1}^{p} x_i^{(k+1)} \alpha_i(t) + \mu \sum_{i=1}^{m} y_i^{(k+1)} \gamma_i(t) + G_k(t),$$

where $x_i^{(k+1)}, i = 1, \ldots, p$ and $y_i^{(k+1)}, i = 1, \ldots, m$ represent the solution of the linear system

$$(2.12) \quad \begin{cases} X^{(k+1)} = \lambda U X^{(k+1)} + \mu V Y^{(k+1)} + \theta^{(k+1)} \\ Y^{(k+1)} = \lambda W^{(k+1)} X^{(k+1)} + \mu T^{(k+1)} Y^{(k+1)} + \varepsilon^{(k+1)} \end{cases}$$

where

$$U = (a_{ij})_{1 \leq i, j \leq p}, \qquad V = (b_{ji})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq m}},$$

$$W^{(k+1)} = (A_{ji})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq m}}, \qquad T^{(k+1)} = (B_{ji})_{1 \leq i, j \leq m},$$

$$X^{(k+1)} = \left( x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_m^{(k+1)} \right)^T,$$

$$Y^{(k+1)} = \left( y_1^{(k+1)}, y_2^{(k+1)}, \ldots, y_p^{(k+1)} \right)^T,$$

$$\theta^{(k+1)} = \left( \theta_1^{(k+1)}, \theta_2^{(k+1)}, \ldots, \theta_p^{(k+1)} \right)^T,$$

$$\varepsilon^{(k+1)} = \left( \varepsilon_1^{(k+1)}, \varepsilon_2^{(k+1)}, \ldots, \varepsilon_m^{(k+1)} \right)^T.$$

It can be easily seen that under the conditions of Theorem 3.1 the sequences $(X^k)_{k \geq 0}$ and $(Y^k)_{k \geq 0}$ converge.

## 3. THE APPROXIMATION OF THE EIGENPAIRS OF MATRICES

Denote $V = \mathbb{K}^n$ and let $A \in \mathbb{K}^{n \times n}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As we have seen in [5], for computing the eigenpairs of $A$ we may consider a mapping $G : V \to \mathbb{K}$ with $G(0) \neq 1$. The eigenvalues and eigenvectors are the solution of the nonlinear system

$$(3.1) \quad F(x) = \begin{pmatrix} Av - \lambda v \\ G(v) - 1 \end{pmatrix} = 0,$$

where $x = \begin{pmatrix} v \\ \lambda \end{pmatrix} \in V \times \mathbb{K} = \mathbb{K}^{n+1}$. Denoting $v = \left( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \right)$ and $\lambda = x^{(n+1)}$ then the first $n$ components of $F$, $F_i, i = 1, \ldots, n$, are given by

$$F_i(x) = a_{i1} x^{(1)} + \cdots + a_{i,i-1} x^{(i-1)} + \left( a_{ii} - x^{(n+1)} \right) x^{(i)} + a_{i,i+1} x^{(i+1)} + \cdots + a_{in} x^{(n)}.$$

If we take $G$ as

$$G(v) = a\|v\|_2^2$$

for some fixed $a \in \mathbb{R}$ then

$$F_{n+1}(x) = a\left(\left(x^{(1)}\right)^2 + \cdots + \left(x^{(n)}\right)^2\right) - 1.$$

The matrices associated to the order divided differences of $F$ on the points $x_1, x_2 \in \mathbb{K}^{n+1}$ are

$$[x_1, x_2; F] = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1n} & a^1_{1,n+1} \\ a_{21} & b_{22} & \cdots & a_{2n} & a^1_{2,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_{nn} & a^1_{n,n+1} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & 0 \end{pmatrix}$$

where $b_{ii} = a_{ii} - \frac{1}{2}\left(x_2^{(n+1)} + x_1^{(n+1)}\right)$, $a^1_{i,n+1} = -\frac{1}{2}\left(x_2^{(i)} + x_1^{(i)}\right)$ and $a_{n+1,i} = a\left(x_1^{(i)} + x_2^{(i)}\right)$ for $i = 1, \ldots, n$.

The second order divided differences of $F$ on $x_1, x_2, x_3$ are given by

$$[x_1, x_2, x_3; F] = \begin{pmatrix} -\frac{1}{2}k^{(n+1)} & 0 & \cdots & 0 & -\frac{1}{2}k^{(1)} \\ 0 & -\frac{1}{2}k^{(n+1)} & \cdots & 0 & -\frac{1}{2}k^{(2)} \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2}k^{(n+1)} & -\frac{1}{2}k^{(n)} \\ ak^{(1)} & ak^{(2)} & \cdots & ak^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix}$$

for all $h, k \in \mathbb{K}^{n+1}$.

We shall consider the max norm on $V$ and taking $\|x\| = \max\{\|v\|_\infty, |\lambda|\}$ for all $x = \begin{pmatrix} v \\ \lambda \end{pmatrix} \in \mathbb{K}^{n+1}$ we are led to the max norm on $\mathbb{K}^{n+1}$. It can be easily verified that

$$\|[x_1, x_2, x_3; F]\|_\infty \le \max\{1, n|a|\}.$$

Let $x_0, x_1 \in \mathbb{K}^{n+1}$ be such that $[x_0, x_1; F]$ is nonsingular. Denote $\hat{b}_0 = \|[x_0, x_1; F]^{-1}\|_\infty$, $\hat{d}_0 = \|x_0 - x_1\|_\infty$, $\hat{\alpha} = \max\{1, n|a|\}$ and $\overline{B}_{\hat{r}} = \left\{x \in \mathbb{K}^{n+1} \mid \|x - x_0\|_\infty \le \hat{r}\right\}$. Applying theorem 4.1 from [5] we get

THEOREM 3.1. *Assume that the matrix* $[x_0, x_1; F]$ *is nonsingular and the numbers* $\hat{b}_0$, $\hat{d}$, $\hat{\alpha}$ *and* $\hat{r}$ *satisfy:*

a) $\hat{b}_0\left(2\hat{r} + \hat{d}_0\right) = \hat{q} < 1;$

b) $\varepsilon_0 = b^2\|F(x_0)\|_\infty < 1$, $b^2\|F(x_0)\|_\infty \le \varepsilon_0^l$ *where* $b = \dfrac{\hat{b}_0}{1 - \hat{q}}$ *and* $l = \dfrac{1 + \sqrt{5}}{2}$;

c) $\dfrac{\varepsilon_0^l}{b(1 - \varepsilon_0)} + \hat{d}_0 \le \hat{r}.$

*Then the sequence* $(x_k)_{k \ge 0}$ *given by the iterations* $[x_{k-1}, x_k; F](x_{k+1} - x_k) + F(x_k) = 0$, $k = 1, 2, \ldots$ *lie in the ball* $\overline{B}_{\hat{r}}(x_0)$ *and converges. Denoting* $x^* = \lim\limits_{k \to \infty} x_k$, *then* $F(x^*) = 0$ *and the following estimations hold*

$$\|x^* - x_k\|_\infty \le \frac{\varepsilon_0^{l^k}}{b(1 - \varepsilon_0^{l^k})}, \quad k = 2, 3, \ldots$$

## 4. NUMERICAL EXAMPLES

We shall consider two test matrices[1] in order to study the behavior of the chord method for approximating the eigenpairs. The programs were written in Matlab[2] and were run on a PC.

PORES1 MATRIX. This matrix arise from oil reservoir simulation. It is real, unsymmetric, of dimension 30 and has 20 real eigenvalues. We have chosen to study the largest eigenvalue $\lambda^* = -1.8363e + 1$. The initial approximations were taken $\lambda_0 = \lambda^* + 0.5$ and $\lambda_1 = \lambda^* + 0.25$; for the initial vector $v_0$ we perturbed the solution $v^*$ (computed by Matlab and then properly scaled to fulfill the norming equation) with random vectors having the components uniformly distributed on $(-\varepsilon, \varepsilon)$, $\varepsilon = 0.2$; for the vector $v_1$ we halved the perturbation. The following results are typical for the runs made (we have considered here the same vector $\varepsilon$ for the four initial approximations), where for choice I we took in $G$ $a = \dfrac{1}{2}$, while for choice II $a = \dfrac{1}{2n}$.

FIDAP002 MATRIX. This real symmetric matrix of dimension $n = 441$ arise from finite element modeling. Its eigenvalues are all simple and range from

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[1] These matrices are available from MatrixMarket at the following address: http://math.nist.gov/MatrixMarket/.

[2] MATLAB is a registered trademark of the MathWorks, Inc.

*Table 1*

The chord method for Pores1

| k | Choice I ‖x* − x_k‖ | ‖F(x_k)‖ | Choice II ‖x* − x_k‖ | ‖F(x_k)‖ |
|---|---|---|---|---|
| 0 | 8.5333e−01 | 1.7938e+06 | 8.5333e−01 | 1.7938e+06 |
| 1 | 4.2667e−01 | 8.9691e+05 | 4.2667e−01 | 8.9690e+05 |
| 2 | 7.8601e−02 | 2.0609e−01 | 1.6245e−02 | 1.7221e−01 |
| 3 | 6.6831e−03 | 9.5180e−03 | 2.9086e−04 | 1.9677e−03 |
| 4 | 1.7647e−04 | 2.5007e−04 | 2.9783e−07 | 2.1405e−06 |
| 5 | 2.3706e−07 | 3.3458e−07 | 7.5862e−11 | 9.1430e−10 |
| 6 | 4.6624e−10 | 8.2317e−10 | | |

$-7 \cdot 10^8$ to $3 \cdot 10^6$. We have chosen to study the smallest eigenvalue, which is well separated. The initial approximations were taken $\lambda_0 = \lambda^* + 10^2 = -6.9996 \cdot 10^8 + 100$, resp. $\lambda_1 = \lambda^* + 10$; for the initial vector $v_0$ we perturbed the solution $v^*$ with random vectors having the components uniformly distributed on $(-\varepsilon, \varepsilon)$, $\varepsilon = 0.5$; for the vector $v_1$ we halved the perturbation. The following results are typical for the runs made (we have considered a common vector $\varepsilon$).

*Table 2*

The secant method for Fidap002

| k | Choice I ‖x* − x_k‖ | ‖F(x_k)‖ | Choice II ‖x* − x_k‖ | ‖F(x_k)‖ |
|---|---|---|---|---|
| 0 | 1.0003e+2 | 1.7467e+9 | 2.5107e+0 | 1.7467e+9 |
| 1 | 1.0078e+1 | 8.7335e+8 | 1.2553e+0 | 8.7335e+8 |
| 2 | 3.4853e+1 | 1.6462e+2 | 5.2877e−2 | 3.5641e−3 |
| 3 | 2.1368e+0 | 2.3373e+1 | 6.1804e−5 | 4.9048e−6 |
| 4 | 4.4761e−1 | 6.3521e−1 | 5.9858e−7 | 4.4309e−6 |
| 5 | 6.5214e−3 | 9.2238e−3 | | |
| 6 | 1.5617e−5 | 2.2961e−5 | | |
| 7 | 5.9605e−7 | 8.5647e−8 | | |

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