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CUBIC TRIGONOMETRIC SPLINE FUNCTIONS OF INTERPOLATION AND APPLICATIONS

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1. INTRODUCTION

Trigonometric spline functions were first studied by Schoenberg [9]. It was shown that any trigonometric spline could be expressed as a linear combination of certain trigonometric B-splines. Lyche and Winther [8], studied a trigonometric analog of divided differences and then they derived a stable recurrence relation for trigonometric B-splines. One method to obtain spline functions of interpolation is to use B-spline functions. This approach follows, as an example, the steps presented in Schumaker [10]. Using the algebraic theory of the classical polynomials, an alternative method for obtaining cubic spline functions of interpolation was deeply analysed by Ahlberg, Nilson and Walsh [1].

In this paper, using the algebraic theory of trigonometric polynomials, developed in [6], and the ideas from [1], we present two constructions of the cubic trigonometric spline functions of interpolation.

Next section presents the basic results from the algebraic theory of trigonometric polynomials that are used in the sections 3 and 4.

The section 3 is dedicated to the study of the existence and the uniqueness of the cubic trigonometric spline function of interpolation. Two approaches are presented. The first one is based on the computation of the derivative of the cubic spline function in the interpolation knots, while, the second approach uses the second derivative of the cubic spline function in the interpolation knots.

In the fourth section, firstly, the Hermite trigonometric quadrature formula [7] is reminded, then a quasi-Hermite trigonometric quadrature formula is presented. The results from §3 and these trigonometric quadrature formulas are used in order to obtain another two quadrature formulas based on trigonometric spline functions of interpolation. These final quadrature formulas can be used when the function under integration is a continuous one, while the trigonometric Hermite and quasi-Hermite quadrature formulas are valid only for derivable functions.

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Equidistant and nonequidistant interpolation knots are considered in numerical integration by trigonometric spline interpolation.

2. PRELIMINARIES

Let K be a field which has characteristic zero, $n \geq 1$, and the set defined by:

$$\tilde{K}[X] := \{u(X) \,|\, (u(X) = \sum_{i=1}^{n} \left(a_i \cos\left(\frac{2i-1}{2}X\right) + b_i \sin\left(\frac{2i-1}{2}X\right)\right)) \,\&$$

$$\&(\forall i)(i \in \overline{1,n})(a_i, b_i \in K)\}$$

(resp.

$$\tilde{\tilde{K}}[X] := \{u(X) \,|\, (u(X) = a_0 + \sum_{i=1}^{n} (a_i \cos(iX) + b_i \sin(iX))) \,\&$$

$$\& (a_0 \in K) \,\& (\forall i)(i \in \overline{1,n})(a_i, b_i \in K)\}.$$

The elements of the set $\tilde{K}[X]$ (resp. $\tilde{\tilde{K}}[X]$) will be called trigonometric polynomials in variable X with odd (resp. even) degree and coefficients in the field K.

If $u(x) \in \tilde{K}[X] \cup \tilde{\tilde{K}}[X]$ and $a_n \neq 0$ or $b_n \neq 0$ then it is said that the polynomial u has the degree n. In the following, the degree of a polynomial u will be denoted by ∂u.

Let $S_1(K) := \{(x,y) \,|\, ((x,y) \in K^2) \,\& (x^2 + y^2 = 1)\}$. If $\theta \in S_1(K)$ let be $\cos\theta := pr_1\theta$ and $\sin\theta := pr_2\theta$. Let $p \in \tilde{K}[X] \cup \tilde{\tilde{K}}[X]$, the element $\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \in S_1(K)$ is a root of the polynomial p, with the order of multiplicity k (a k-fold root), if $p(X) = \left(\sin\frac{X-\theta}{2}\right)^k p_1(X)$, where $p_1 \in \tilde{K}[X] \cup \tilde{\tilde{K}}[X]$ and $\sin\frac{X-\theta}{2}$ is not a divisor of p_1. Here $\sin\frac{X-\theta}{2}$ stands for $\cos\frac{\theta}{2}\sin\frac{X}{2} - \sin\frac{\theta}{2}\cos\frac{X}{2}$.

It is known (Cor. 4 [6]) that $\frac{\theta}{2} \in S_1(K)$ is a k-fold root ($k \geq 1$) of the trigonometric polynomial $u \in \tilde{K}[X] \cup \tilde{\tilde{K}}[X]$ if and only if

$$u(\theta/2) = Du(\theta/2) = \ldots = D^{k-1}u(\theta/2) = 0 \text{ and } D^k u(\theta/2) \neq 0.$$

Two elements $\frac{\theta_1}{2} \in S_1(K)$ and $\frac{\theta_2}{2} \in S_1(K)$ will be called distinct if

$\frac{\theta_1}{2} \neq \frac{\theta_2}{2}$ and $\frac{\theta_1}{2} \neq \left(-\cos\frac{\theta_2}{2}, -\sin\frac{\theta_2}{2}\right)$.

The following theorem, is an important result, with theoretical and practical applications in the theory of trigonometric interpolation. See also [2] and [4].

THEOREM 2.1 (Th. 6 [6]). *Let be* $\left(\cos\frac{\theta_i}{2}, \sin\frac{\theta_i}{2}\right)_{i\in\overline{0,m}}$, *$m$+1 distinct elements belonging to* $S_1(K)$, $(n_0, n_1, \ldots, n_m)^T$ *a vector from* $(N^*)^{m+1}$ *such that* $n+1 := \sum_{j=0}^{m} n_j$.

Also, let $f := (f_0^{(0)}, \ldots, f_0^{(\mu_0)}, \ldots, f_j^{(0)}, \ldots, f_j^{(\mu_j)}, \ldots, f_m^{(\mu_m)})^T \in K^{n+1}$, *where* $\mu_j := n_j - 1$ *for all* $j \in \overline{0,m}$. *There exists a unique trigonometric polynomial (called* the Hermite interpolation trigonometric polinomial*)* $u \in \tilde{K}[X] \cup \tilde{\tilde{K}}[X]$ *with* $\partial u \leq n$ *such that*

$$(D^i u)\left(\frac{\theta_j}{2}\right) = f_j^i, \quad 0 \leq j \leq m, \quad 0 \leq i \leq \mu_j.$$

Moreover, if $n = 2k$ (resp. $n = 2k-1$) there exists a base (called the Hermite base*)* $(H_{ji})_{j\in\overline{0,m},\, i\in\overline{0,\mu_j}}$ *in* $\{p \,|\, p \in \tilde{\tilde{K}}[X]) \,\& (\partial p \leq 2k)\}$ *(resp.* $\{q \,|\, (q \in \tilde{K}[X]) \,\&$ $\& (\partial p \leq 2k-1)\}$*) such that*

$$(D^l H_{ji})\left(\frac{\theta_t}{2}\right) = \delta_{jt}\delta_{il}, \quad 0 \leq j, t \leq m, 0 \leq i \leq \mu_j, 0 \leq l \leq \mu_t$$

and

$$u(x) = \sum_{j=0}^{m}\sum_{i=0}^{\mu_j} f_j^{(i)} H_{ji}(x). \tag{1}$$

If $m = 0$ the above base will be called the Taylor base. The following results, presented in [6] (prop. 7), show how we can obtain the elements of the Taylor base.

If $n_0 = 2k+1$ then the Taylor base denoted by $(T_{2k,j})_{j\in\overline{0,2k}}$ satisfies:

$$T_{2k,2k}(x) = \frac{2^{2k}}{(2k)!}\left(\sin\frac{x-\theta_0}{2}\right)^{2k} \text{ and}$$

$$T_{2k,j-1}(x) = DT_{2k,j}(x) + (L_{2k+1}q_j)(x_0)T_{2k,2k}(x),$$

where $j \in \overline{1,2k}$, $q_j(x) := \frac{(x-x_0)^j}{j!}$, $x_0 \in K$ and $L_{2k+1} := D\prod_{i=1}^{k}(D^2 + i^2 I)$.

Similarly, if $n_0 = 2k$ then the Taylor base denoted by $(T_{2k-1,j})_{j\in\overline{0,2k-1}}$ satisfies:

$$T_{2k-1,2k-1}(x) = \frac{2^{2k-1}}{(2k-1)!}\left(\sin\frac{x-\theta_0}{2}\right)^{2k-1} \text{ and}$$

$$T_{2k-1,j-1}(x) = DT_{2k-1,j}(x) + (L_{2k}q_j)(x_0)T_{2k-1,2k-1}(x),$$

where $j\in\overline{1,2k-1}$, $q_j(x) := \frac{(x-x_0)^j}{j!}$, $x_0 \in K$ and $L_{2k} := \prod_{i=1}^{k}\left(D^2 + \left(\frac{2i-1}{2}\right)^2 I\right)$.

The elements of the Hermite base are given by the following theorem.

THEOREM 2.2 (Th. 9 [6]). *Let* $\left(\cos\frac{\theta_i}{2}, \sin\frac{\theta_i}{2}\right)_{i\in\overline{0,m}}$ *be m+1 distinct elements belonging to* $S_1(K)$, $(n_0, n_1, \ldots, n_m)^T$ *a vector from* $(N^*)^{m+1}$ *such that* $n+1 := \sum_{j=0}^{m} n_j$ *and* $f := (f_0^{(0)}, \ldots, f_0^{(\mu_0)}, \ldots, f_j^{(0)}, \ldots, f_j^{(\mu_j)}, \ldots, f_m^{(\mu_m)})^T \in K^{n+1}$, $(\mu_j := n_j - 1$ *for all* $j\in\overline{0,m}$ *). If* $(H_{ji})_{j\in\overline{0,m},\, i\in\overline{0,\mu_j}}$ *is the Hermite base and u is the Hermite interpolation trigonometric polynomial, for these data, then*

$$H_{ji}(x) = \left(\sin\frac{x-\theta_j}{2}\right)^i g_j(x)\sum_{p=0}^{\mu_j - i}\left(D^p\frac{u_i^{(j)}}{g_j}\right)\left(\frac{\theta_j}{2}\right)T^{(j)}_{\mu_j-i,p}(x)$$

where $g_j(x) := \prod_{l\neq j}^{m}\left(\sin\frac{x-\theta_l}{2}\right)^{n_l}$, $\left(T^{(j)}_{\mu_j-i,p}\right)_{p\in\overline{0,\mu_j-i}}$ *is the Taylor base which corresponds to the point* $\left(\cos\frac{\theta_j}{2}, \sin\frac{\theta_j}{2}\right)$ *and* $T^{(j)}_{n,i}(x) = \left(\sin\frac{x-\theta_j}{2}\right)^i u_i^{(j)}(x)$,

and

$$u(x) = \sum_{j=0}^{m} g_j(x)\sum_{i=0}^{\mu_j} f_j^i\left(\sin\frac{x-\theta_j}{2}\right)^i\sum_{p=0}^{\mu_j-i}\left(D^p\frac{u_i^{(j)}}{g_j}\right)\left(\frac{\theta_j}{2}\right)T^{(j)}_{\mu_j-i,p}(x).$$

3. CUBIC TRIGONOMETRIC SPLINE FUNCTIONS OF INTERPOLATION

Let $[a, b] \subset R$, and $\Delta : \{x_i \mid i\in\overline{0,\nu}\}$ a division with $\nu + 1$ points of the interval $[a, b]$, such that $a = x_0 < x_1 < \ldots < x_{\nu-1} < x_\nu = b$. Also let $f := (f_i)_{i\in\overline{0,\nu}}$, $\nu + 1$ real numbers.

DEFINITION 3.1. *The function* $s_{\Delta,f} \in C^{(2)}([a,b]; R)$ *which satisfies:*

(2) $$(\forall i)(i\in\overline{0,\nu})(s_{\Delta,f}(x_i) = f_i) \text{ and}$$

(3) $$(\forall i)(i\in\overline{0,\nu-1})\left((s_{\Delta,f}\big|_{[x_{i-1},x_i]} \in \tilde{R}[X])\,\&\,(\partial s_{\Delta,f}\big|_{[x_{i-1},x_i]} \leq 3)\right)$$

will be called cubic trigonometric spline function with respect to Δ, *which interpolates the data* f_i *in the knots* x_i. *Such a function will be called periodic with period* $b - a$ *if the following relation holds:*

(4) $$s^{(p)}_{\Delta,f}(a+) = s^{(p)}_{\Delta,f}(b-) \text{ for } p\in\overline{0,2}.$$

The following lemma can be obtained using the theorem 2.2.

LEMMA 3.1. *Let* α, β *be two different real numbers,* $H[\alpha, \beta, 0]$, $H[\alpha, \beta, 1]$, $H[\beta, \alpha, 0]$ *and* $H[\beta, \alpha, 1]$ *the Hermite base from* $\{u \mid (u\in\tilde{R}[X])\,\&\,(\partial u \leq 3)\}$ *attached to the knots* α, β *with multiplicities* $n_\alpha = n_\beta = 2$. *Then*

$$H^{(k)}[\alpha, \beta, i](\gamma) = \delta_{\alpha\gamma}\cdot\delta_{ki} \text{ and}$$
$$H^{(k)}[\beta, \alpha, i](\gamma) = \delta_{\beta\gamma}\cdot\delta_{ki},\ \gamma\in\{\alpha, \beta\} \text{ and } k, i \in \{0, 1\}.$$

Moreover

$$H[\alpha,\beta,0](x) = \frac{\sin^2\frac{x-\beta}{2}}{\sin^3\frac{x-\beta}{2}}\left[\sin\frac{\beta-\alpha}{2}\cos\frac{x-\alpha}{2} + 2\cos\frac{\beta-\alpha}{2}\sin\frac{x-\alpha}{2}\right]$$

(5) $$= \frac{1}{8\sin^3\frac{\beta-\alpha}{2}}\left[6\sin\frac{x+\beta-2\alpha}{2} + 3\sin\frac{x-\beta}{2} - 3\sin\frac{3x-\beta-2\alpha}{2} + 3\sin\frac{x+2\alpha-3\beta}{2} - \sin\frac{3(x-\beta)}{2}\right]$$

and

$$H[\alpha,\beta,1](x) = \frac{2\sin\frac{x-\alpha}{2}\sin^2\frac{x-\beta}{2}}{\sin^2\frac{\beta-\alpha}{2}}$$

(6) $$= \frac{1}{2\sin^2\frac{\beta-\alpha}{2}}\left[2\sin\frac{x-\alpha}{2} - \sin\frac{3x-2\beta-\alpha}{2} + \sin\frac{x+\alpha-2\beta}{2}\right].$$

The polynomials $H[\beta, \alpha, 0]$ *and* $H[\beta, \alpha, 1]$ *are obtained from the above relations by interchanging* α *and* β.

Now, we are ready to present the first kind formula for the trigonometric cubic spline function of interpolation.

THEOREM 3.1. *Let* $\Delta := \{x_i | i \in \overline{0,\nu}\}$ *be a division of the interval* $[a, b]$ *with* $b - a < 2\pi$ *and* $f : (f_i)_{i\in\overline{0,\nu}}$ *a vector from* $R^{\nu+1}$. *If a cubic trigonometric spline function* $\tilde{s}_{\Delta,f}$ *interpolates the data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$, *then* $m_i := \tilde{s}''_{\Delta,f}(x_i), i \in \overline{0,\nu}$ *satisfy the following rectangular system of linear equa- tions:*

(7) $$\rho_{i+1}m_i + 2m_{i+1} + \lambda_{i+1}m_{i+2} = d_{i+1}, \quad i \in \overline{0, \nu-2},$$

where

$$h_i := x_{i+1} - x_i, \quad \rho_{i+1} := \left(\sin\frac{h_{i+1}}{2}\right) \Big/ \sin\frac{h_{i+1}+h_i}{2},$$

$$\lambda_{i+1} := \left(\sin\frac{h_i}{2}\right) \Big/ \sin\frac{h_{i+1}+h_i}{2},$$

and

$$d_{i+1} := \frac{3}{2}\cdot\frac{1}{\sin\frac{h_{i+1}+h_i}{2}}\left[\sin\frac{h_{i+1}}{2}\,\frac{f_{i+1} - f_i\cos\frac{h_i}{2}}{\sin\frac{h_i}{2}} + \sin\frac{h_i}{2}\,\frac{f_{i+2}\cos\frac{h_{i+1}}{2} - f_{i+1}}{\sin\frac{h_{i+1}}{2}}\right].$$

Proof. Let $i \in \overline{0,\nu-2}$, $j \in \{i, i+1\}$, and $p_j := \tilde{s}_{\Delta,f}\big|_{[x_j,x_{j+1}]}$, $p_j \in \tilde{R}[X]$ with $\partial p_i \le 3$. It follows from relation (1) that

(8) $$\begin{aligned} p_j(x) &= f_j H[x_j, x_{j+1}, 0](x) + m_j H[x_j, x_{j+1}, 1](x) + \\ &+ f_{j+1} H[x_{j+1}, x_j, 0](x) + m_{j+1} H[x_{j+1}, x_j, 1](x). \end{aligned}$$

Using lemma 3.1 for $\alpha := x_j$ and $\beta := x_{j+1}$ and considering $h_j := x_{j+1} - x_j$, after some simple computations, we obtain:

(9) $$\begin{aligned} H''[x_j, x_{j+1}, 0](x) = \frac{1}{8\sin^3\frac{x_{j+1}-x_j}{2}}\Bigg[&\frac{27}{4}\sin\frac{3x - x_{j+1} - 2x_j}{2} + \\ &+ \frac{9}{4}\sin\frac{3(x - x_{j+1})}{2} - \frac{3}{2}\sin\frac{x + x_{j+1} - 2x_j}{2} - \\ &- \frac{3}{4}\sin\frac{x - x_{j+1}}{2} - \frac{3}{4}\sin\frac{x - 3x_{j+1} + 2x_j}{2}\Bigg] \end{aligned}$$

and

(10) $$\begin{aligned} H''[x_j, x_{j+1}, 1](x) = \frac{1}{8\sin^2\frac{x_{j+1}-x_j}{2}}\Bigg[&9\sin\frac{3x - 2x_{j+1} - x_j}{2} - \\ &- \sin\frac{x - 2x_{j+1} + x_j}{2} - 2\sin\frac{x - x_j}{2}\Bigg]. \end{aligned}$$

Therefore

(11) $$H''[x_i, x_{i+1}, 0](x_{i+1}) = \frac{3}{2}\cos\frac{h_i}{2} \Big/ \sin^2\frac{h_i}{2}$$

and

(12) $$H''[x_i, x_{i+1}, 1](x_{i+1}) = 1 \Big/ \sin\frac{h_i}{2}.$$

Interchanging x_j and x_{j+1} in the relations (9) and (10) we obtain:

(13) $$H''[x_i, x_{i+1}, 0](x_{i+1}) = -\frac{3}{2}\frac{1}{\sin^2\frac{h_i}{2}} + \frac{3}{4}$$

and

(14) $$H''[x_i, x_{i+1}, 1](x_{i+1}) = 2\frac{\cos\frac{h_i}{2}}{\sin\frac{h_i}{2}}.$$

It follows that

(15) $$\begin{aligned} p''_i(x_{i+1}) = \frac{3}{2} f_i \frac{\cos\frac{h_i}{2}}{\sin^2\frac{h_i}{2}} + m_i \frac{1}{\sin\frac{h_i}{2}} + \\ + f_{i+1}\left(-\frac{3}{2}\frac{1}{\sin^2\frac{h_i}{2}} + \frac{3}{4}\right) + m_{i+1}\frac{2\cos\frac{h_i}{2}}{\sin\frac{h_i}{2}}. \end{aligned}$$

In the same way we obtain:

(16) $$H''[x_{i+1}, x_{i+2}, 0](x_{i+1}) = -\frac{3}{2}\frac{1}{\sin^2\frac{h_{i+1}}{2}} + \frac{3}{4},$$

(17) $$H''[x_{i+1}, x_{i+2}, 1](x_{i+1}) = -2\cos\frac{h_{i+1}}{2} \Big/ \sin\frac{h_{i+1}}{2},$$

(18) $$H''[x_{i+2}, x_{i+1}, 0](x_{i+1}) = \frac{3}{2}\cos\frac{h_{i+1}}{2} \Big/ \sin^2\frac{h_{i+1}}{2}$$

and

$$H''[x_{i+2}, x_{i+1}, 1](x_{i+1}) = -\frac{1}{\sin\frac{h_{i+1}}{2}}. \tag{19}$$

Therefore

$$p''_{i+1}(x_{i+1}) = \frac{3}{2} f_{i+2} \frac{\cos\frac{h_{i+1}}{2}}{\sin^2\frac{h_{i+1}}{2}} - m_{i+2}\frac{1}{\sin\frac{h_{i+1}}{2}} + \tag{20}$$
$$+ f_{i+1}\left(-\frac{3}{2}\frac{1}{\sin^2\frac{h_{i+1}}{2}} + \frac{3}{4}\right) - m_{i+1}\frac{2\cos\frac{h_{i+1}}{2}}{\sin\frac{h_{i+1}}{2}}.$$

The function $\tilde{s}_{\Delta,f}$ will be a cubic trigonometric spline if and only if $p''_i(x_{i+1}) = p''_{i+1}(x_{i+1})$ for $i \in \overline{0, \nu - 2}$, which is equivalent with (7).

COROLLARY 3.1. *For every* $i \in \overline{0, \nu-2}$, $\rho_{i+1} \neq 0$ *and there exists a cubic trigonometric spline function which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$.

Proof. We remark that the matrix of the system (7) has full rank $\nu - 1$. Hence, there exists a cubic trigonometric spline function which interpolates data $(f_i)_{i\in\overline{0,\nu}}$ in the knots $(x_i)_{i\in\overline{0,\nu}}$.

Such a function, on the interval $[x_i, x_{i+1}]$ (with $i \in \overline{0, \nu - 1}$), is given by relation (8).

REMARK 3.1. *Let* T *be the matrix of the system (7),* $d := (d_1, d_2, \ldots, d_{\nu-1})^t$ *and* T^+ *the pseudoinverse of matrix* T *[3], [4]. Then* $m := (m_0, m_1, \ldots, m_\nu)^t := T^+d$ *determines a cubic trigonometric spline function* $\tilde{s}_{\Delta,f}$, *such that for every cubic trigonometric spline function* $s_{\Delta,f}$ *which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$, *we have*

$$\left\|(\tilde{s}'_{\Delta,f}(x_i))_{i\in\overline{0,\nu}}\right\|_2 \le \left\|(s'_{\Delta,f}(x_i))_{i\in\overline{0,\nu}}\right\|_2 .$$

For the nonperiodic case, if we add to the relation (7), the boundary relations

$$\begin{cases} 2m_0 + \lambda_0\, m_1 = d_0 \\ \rho_\nu\, m_{\nu-1} + 2m_\nu = d_\nu, \end{cases} \tag{21}$$

the obtained system will have a unique solution if its tridiagonal matrix is diagonal dominant ($2 > |\lambda_0|$, $2 > |\rho_\nu|$ and $2 > |\rho_{i+1}| + |\lambda_{i+1}|$ for $i \in \overline{0, \nu-2}$). Therefore, in this case, there exists a unique cubic trigonometric spline which interpolates data $(f_i)_{i\in\overline{0,\nu}}$ in the knots $(x_i)_{i\in\overline{0,\nu}}$. Also, other "end conditions" can be specified.

For the periodic spline ($f_\nu = f_0$ and $m_\nu = m_0$), we require that (7) be valid for $i = \nu - 1$ as well. We will obtain a tridiagonal system, and the following corollary is valid.

COROLLARY 3.2. *With the above notations, if* $|\lambda_i| + |\rho_i| < 2$ *for* $i \in \overline{0, \nu}$, *where* $h_\nu := h_0$ *is used to obtain* λ_ν *and* ρ_ν, *then there exists a unique periodic cubic trigonometric spline which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$.

In order to derive a second kind formula for the trigonometric cubic spline function of interpolation, we need the following lemma.

LEMMA 3.2. *Let* α, β *be two different real numbers. The space* $\{u \mid (u \in \tilde{R}[X]) \,\&\, (\partial u \le 3)\}$ *admites a base* $B[\alpha, \beta, 0]$, $B[\alpha, \beta, 2]$, $B[\beta, \alpha, 0]$, *and* $B[\beta, \alpha, 2]$ *which satisfies*:

$$\begin{aligned} &B^{(k)}[\alpha, \beta, i](\gamma) = \delta_{\alpha\gamma}\cdot\delta_{ki} \\ &B^{(k)}[\beta, \alpha, i](\gamma) = \delta_{\beta\gamma}\cdot\delta_{ki}, \quad \gamma \in \{\alpha, \beta\} \text{ and } k, i \in \{0, 2\}. \end{aligned} \tag{22}$$

Moreover

$$B[\alpha, \beta, 0](x) = \frac{9}{8\sin\frac{3(\beta-\alpha)}{2}}\left[\frac{1}{9}\sin\frac{3(x-\beta)}{2} - \right. \tag{23}$$
$$\left. -\sin\frac{x+2\alpha-3\beta}{2} - \sin\frac{x-\beta}{2} - \sin\frac{x-2\alpha+\beta}{2}\right]$$

and

$$B[\alpha, \beta, 2](x) = \frac{1}{2\sin\frac{3(\beta-\alpha)}{2}}\left[\sin\frac{3(x-\beta)}{2} - \right. \tag{24}$$
$$\left. -\sin\frac{x+2\alpha-3\beta}{2} - \sin\frac{x-\beta}{2} - \sin\frac{x-2\alpha+\beta}{2}\right]$$

The polynomials $B[\alpha, \beta, 0](x)$ *and* $B[\alpha, \beta, 2](x)$ *are obtained from the above relations by interchanging* α *and* β.

THEOREM 3.2. *Let* $\Delta := \{x_i \mid h_i := x_{i+1} - x_i < \frac{2\pi}{3}, i \in \overline{0, \nu-1}\}$ *be a division of the interval* $[a, b]$ *with* $b - a < 2\pi$ *and* $f := (f_i)_{i\in\overline{0,\nu}}$ *a vector from* $R^{\nu+1}$. *If a* *cubic trigonometric spline function* $\tilde{s}_{\Delta,f}$ *interpolates the data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$ *then* $M_i := \tilde{s}^{(2)}_{\Delta,f}(x_i)$, $i\in\overline{0,\nu}$ *satisfy the linear system*:

$$\tilde{\rho}_{i+1}M_i + \tilde{\delta}_{i+1}M_{i+1} + \tilde{\lambda}_{i+1}M_{i+2} = \tilde{d}_{i+1}, \quad i\in\overline{0,\nu-2}, \tag{25}$$

where

$$\tilde{\rho}_{i+1} := \frac{\sin\frac{h_i}{2}}{1+2\cosh_i},$$

$$\tilde{\delta}_{i+1} := \frac{2\sin\frac{h_i+h_{i+1}}{2}\left[\cos\frac{h_{i+1}-h_i}{2}+2\cos\frac{h_{i+1}+h_i}{2}\right]}{(1+2\cos h_i)(1+2\cos h_{i+1})},$$

$$\tilde{\lambda}_{i+1} := \frac{\sin\frac{h_{i+1}}{2}}{1+2\cos h_{i+1}}$$

and

$$\tilde{d}_{i+1} := f_i\frac{3(1+3\cos h_i)}{8\sin\frac{h_i}{2}(1+2\cos h_i)} - \frac{3}{2}f_{i+1}\left[\frac{\cos^3\frac{h_i}{2}}{\sin\frac{h_i}{2}(1+2\cos h_i)}+\frac{\cos^3\frac{h_{i+1}}{2}}{\sin\frac{h_{i+1}}{2}(1+2\cos h_i)}\right] + f_{i+2}\frac{3(1+3\cos h_{i+1})}{8\sin\frac{h_{i+1}}{2}(1+2\cos h_{i+1})}.$$

Proof. Let $i\in\overline{0,\nu-2}$, $j\in\{i, i+1\}$, and $q_j := \tilde{s}_{\Delta,f}\big|_{[x_j,x_{j+1}]}, q_j\in\tilde{R}[X]$ with $\partial q_j \le 3$. From the lemma 3.2, it follows that

$$\begin{aligned} q_i(x) = {} & f_iB[x_j, x_{j+1}, 0](x) + M_jB[x_j, x_{j+1}, 2] + \\ & + f_{i+1}B[x_{j+1}, x_j, 0](x) + M_{j+1}B[x_{j+1}, x_j, 2]. \end{aligned} \tag{26}$$

The function $\tilde{s}_{\Delta,f}$ will be a cubic trigonometric spline function of interpolation if and only if $q_i'(x_{i+1}) = q_{i+1}'(x_{i+1})$ for $i\in\overline{0,\nu-2}$. After some elementary computations we obtain

$$\begin{aligned} B'[x_i, x_{i+1}, 0](x) = {} & \frac{9}{8\sin\frac{3h_i}{2}}\left[\frac{1}{6}\cos\frac{3(x-x_{i+1})}{2} - \frac{1}{2}\cos\frac{x+2x_i-3x_{i+1}}{2} - \right. \\ & \left. -\frac{1}{2}\cos\frac{x-x_{i+1}}{2} - \frac{1}{2}\cos\frac{x-2x_i+x_{i+1}}{2}\right] \end{aligned} \tag{27}$$

and

$$\begin{aligned} B'[x_i, x_{i+1}, 2](x) = {} & \frac{1}{2\sin\frac{3h_i}{2}}\left[\frac{3}{2}\cos\frac{3(x-x_{i+1})}{2} - \frac{1}{2}\cos\frac{x+2x_i-3x_{i+1}}{2} - \right. \\ & \left. -\frac{1}{2}\cos\frac{x-x_{i+1}}{2} - \frac{1}{2}\cos\frac{x-2x_i+x_{i+1}}{2}\right]. \end{aligned} \tag{28}$$

Hence

$$B'[x_i, x_{i+1}, 0](x_{i+1}) = -\frac{3}{8\sin\frac{3h_i}{2}}(1+3\cos h_i) \tag{29}$$

and

$$B'[x_i, x_{i+1}, 2](x_{i+1}) = \frac{1}{2\sin\frac{3h_i}{2}}(1-\cos h_i). \tag{30}$$

In a similar way, we obtain

$$B'[x_{i+1}, x_i, 0](x_{i+1}) = \frac{3}{8\sin\frac{3h_i}{2}}\left[3\cos\frac{h_i}{2}+\cos\frac{3h_i}{2}\right] \tag{31}$$

and

$$B'[x_{i+1}, x_i, 2](x_{i+1}) = \frac{1}{2\sin\frac{3h_i}{2}}\left[\cos\frac{h_i}{2}-\cos\frac{3h_i}{2}\right]. \tag{32}$$

Therefore

$$\begin{aligned} q_i(x_{i+1}) = {} & -f_i\frac{3}{8\sin\frac{3h_i}{2}}(1+3\cos h_i) + M_i\frac{1}{2\sin\frac{3h_i}{2}}(1-\cos h_i) + \\ & + f_{i+1}\frac{3}{8\sin\frac{3h_i}{2}}\left(3\cos\frac{h_i}{2}+\cos\frac{3h_i}{2}\right) + \\ & + M_{i+1}\frac{1}{2\sin\frac{3h_i}{2}}\left(\cos\frac{h_i}{2}-\cos\frac{3h_i}{2}\right). \end{aligned} \tag{33}$$

For the interval $[x_{i+1}, x_{i+2}]$ we obtain:

$$B'[x_{i+1}, x_{i+2}, 0](x_{i+1}) = -\frac{3}{8\sin\frac{3h_{i+1}}{2}}\left[\cos\frac{3h_{i+1}}{2}+3\cos\frac{h_{i+1}}{2}\right], \tag{34}$$

$$B'[x_{i+1}, x_{i+2}, 2](x_{i+1}) = \frac{1}{2\sin\frac{3h_{i+1}}{2}}\left[\cos\frac{3h_{i+1}}{2}-\cos\frac{h_{i+1}}{2}\right],\tag{35}$$

$$B'[x_{i+2},x_{i+1},0](x_{i+1}) = \frac{3}{8\sin\frac{3h_{i+1}}{2}}[1+3\cos h_{i+1}], \tag{36}$$

and

$$B'[x_{i+2},x_{i+1},2](x_{i+1}) = \frac{1}{2\sin\frac{3h_{i+1}}{2}}[\cos h_{i+1}-1]. \tag{37}$$

Therefore

$$\begin{aligned} q'_{i+1}(x_{i+1}) = & -f_{i+1}\frac{3}{8\sin\frac{3h_{i+1}}{2}}\left[\cos\frac{3h_{i+1}}{2}+3\cos\frac{h_{i+1}}{2}\right]+ \\ & + M_{i+1}\frac{1}{2\sin\frac{3h_{i+1}}{2}}\left[\cos\frac{3h_{i+1}}{2}-\cos\frac{h_{i+1}}{2}\right]+ \\ & + f_{i+2}\frac{3}{8\sin\frac{3h_{i+1}}{2}}[1+3\cos h_{i+1}]+ \\ & + M_{i+2}\frac{1}{2\sin\frac{3h_{i+1}}{2}}[\cos h_{i+1}-1]. \end{aligned} \tag{38}$$

Using the relations (33) and (38) together with the identities $\sin\frac{3x}{2} = \sin\frac{x}{2}(1+2\cos x)$, $\cos\frac{x}{2}-\cos\frac{3x}{2} = 2\sin\frac{x}{2}\sin x$ and $\cos\frac{3x}{2}+3\cos\frac{x}{2} = 4\cos^3\frac{x}{2}$, we obtain the relation (25).

COROLLARY 3.3. *For every* $i\in\overline{0,\nu-2}$, $\tilde{\rho}_{i+1}\neq 0$ *and there exists a cubic trigonometric spline function which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$.

Such a function, in the interval $[x_{i+1},x_{i+2}]$ $(i\in\overline{0,\nu-1})$ *is given by relation* (26).

REMARK 3.2. *Let* $\tilde{T}$ *be the matrix of the system* (25), $\tilde{d} := (\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_{\nu-1})^t$ *and* $\tilde{T}^+$ *the pseudoinverse of matrix* $\tilde{T}$. *Then* $M := (M_0,M_1,\ldots,M_\nu)^t := \tilde{T}^+\tilde{d}$ *determines a cubic trigonometric spline function* $\tilde{s}_{\Delta,f}$, *such that for every cubic trigonometric spline function* $s_{\Delta,f}$ *which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$, *then*

$$\left\|(\tilde{s}''_{\Delta,f}(x_i))_{i\in\overline{0,\nu}}\right\|_2 \le \left\|(s''_{\Delta,f}(x_i))_{i\in\overline{0,\nu}}\right\|_2.$$

For the nonperiodic spline, if we add to the relation (25), the boundary relations

$$\begin{cases}\tilde{\delta}_0M_0+\tilde{\lambda}_0M_1=\tilde{d}_0\\ \tilde{\rho}_\nu M_{\nu-1}+\tilde{\delta}_\nu M_\nu=\tilde{d}_\nu,\end{cases} \tag{39}$$

the obtained system will have a unique solution if its tridiagonal matrix is diagonaly dominant ($|\tilde{\delta}_0|>|\tilde{\lambda}_0|$, $|\tilde{\delta}_\nu|>|\rho_\nu|$ and $|\tilde{\delta}_{i+1}|>|\tilde{\rho}_{i+1}|+|\tilde{\lambda}_{i+1}|$ for $i\in\overline{0,\nu-2}$). Therefore, in this case, there exists a unique cubic trigonometric spline which interpolates data $(f_i)_{i\in\overline{0,\nu}}$ in the knots $(x_i)_{i\in\overline{0,\nu}}$.

For the periodic spline ($f_\nu=f_0$ and $M_\nu=M_0$), we require that (25) be valid for $i=\nu-1$ as well. We will obtain a tridiagonal system, and the following corollary is valid.

COROLLARY 3.4. *With the above notations, if* $|\tilde{\lambda}_i|+|\tilde{\rho}_i|<|\tilde{\delta}_i|$ *for* $i\in\overline{1,\nu}$, *where* $h_\nu := h_0$ *is used to obtain* $\tilde{\lambda}_\nu$ *and* $\tilde{\rho}_\nu$, *then there exists a unique periodic cubic trigonometric spline which interpolates data* $(f_i)_{i\in\overline{0,\nu}}$ *in the knots* $(x_i)_{i\in\overline{0,\nu}}$.

4. NUMERICAL INTEGRATION

The most important applications of the trigonometric spline functions to numerical analysis are numerical differentiation and integration. The results stated by the theorems 3.1 and 3.2, show how trigonometric spline functions of interpolation can be used in approximate differentiation.

This section develops only the application of the trigonometric spline functions of interpolation in approximate integration. In the first part, we present two trigonometric quadrature formulas. The first one is the trigonometric Hermite quadrature formula [7], and will be used to derive the first formula for trigonometric spline numerical integration. In the following, let $\delta(\cdot)$ be the Dirac function.

PROPOSITION 4.1. *Let E be a Banach space, and* $a, b\in R$ $(0<b-a<2\pi)$. *The linear application* $\sigma = a_{00}\delta_a+a_{01}\delta_a^{(1)}+a_{10}\delta_b+a_{11}\delta_b^{(1)}$, *defined on* $C^{(1)}([a,b];E)$, *such that* $\sigma(u)=\int_a^b u(x)\,dx$ *for every* $u\in E\otimes\tilde{R}[X]$ *such that* $\partial u\le 3$, *is given by*

$$\sigma(f) = \frac{2}{3}\frac{\sin\frac{3(b-a)}{4}}{\cos^3\frac{b-a}{4}}(f(a)+f(b))+\frac{4}{3}\tan^2\frac{b-a}{4}(f'(a)-f'(b)), \tag{40}$$

where $f \in C^{(1)}([a,b];E)$.

Moreover, there exists a real constant $M > 0$ such that, for all $a, b \in R$, $0 < b - a < 2\pi$, iff $f \in C^{(4)}([a,b];E)$ then

$$\left\| \int_a^b f(x)\,dx - \sigma(f) \right\| \le M\left(\cos\frac{b-a}{4}\right)^{-3} \sup_{x\in[a,b]} \|(L_4 f)(y)\|(b-a)^5.$$

The second quadrature formula is a quasi-Hermite method for numerical integration. The coefficients of this formula can be obtained in a similar manner as for the Hermite's formula, but using the lemma 3.2 instead of the lemma 3.1. The proof of the following proposition is straightforward and we leave this for the reader.

PROPOSITION 4.2. *Let E be a Banach space, and $a, b \in R$ $(0 < b - a < 2\pi)$. The linear application $\tilde{\sigma} = a_{00}\delta_a + a_{02}\delta_a^{(2)} + a_{10}\delta_b + a_{12}\delta_b^{(2)}$, defined on $C^{(2)}([a,b];E)$, such that $\tilde{\sigma}(u) = \int_a^b u(x)\,dx$ for every $u \in E \otimes \tilde{R}[X]$ such that $\partial u \le 3$, is given by*

$$\tilde{\sigma}(f) = \frac{2}{3}\tan\frac{b-a}{4}\,\frac{13\cos^2\frac{b-a}{4} - 10}{4\cos^2\frac{b-a}{4} - 3}(f(a)+f(b)) - \tag{41}$$
$$-\frac{8}{3}\,\frac{\sin^3\frac{b-a}{4}}{\cos\frac{3(b-a)}{4}}(f''(a)+f''(b)),$$

where $f \in C^{(2)}([a,b];E)$.

The proposition 4.2 will be used to obtain the second formula for trigonometric spline numerical integration.

Let $f \in C([a,b];R)$, $\Delta := \{x_i \mid i \in \overline{0,\nu}\}$ a division of the interval $[a,b]$ and $f_i := f(x_i)$, $i \in \overline{0,\nu}$. The first trigonometric spline quadrature formula results directly from (8) and the proposition 41. We succesively obtain

$$\sigma_\nu(f) = \int_a^b \tilde{s}_\Delta(x)\,dx =$$
$$= \sum_{i=0}^{\nu-1}\int_{x_i}^{x_{i+1}} \tilde{s}_\Delta(x)\,dx = \sum_{i=0}^{\nu-1}\int_{x_i}^{x_{i+1}} p_i(x)\,dx = \tag{42}$$
$$= \frac{2}{3}\sum_{i=0}^{\nu-1}\frac{\sin\frac{3h_i}{4}}{\cos^3\frac{h_i}{4}}(f_i + f_{i+1}) + \frac{4}{3}\sum_{i=0}^{\nu-1}\tan^2\frac{h_i}{4}(m_i - m_{i+1}),$$

where $(m_i)_{i=0}^{\nu}$ satisfy the relations (7) and (21).

When the intervals are of equal length h, then

$$\sigma_\nu(f) = \frac{2}{3}\,\frac{\sin\frac{3h}{4}}{\cos^3\frac{h}{4}}\left(f_0 + 2\sum_{i=1}^{\nu-1} f_i + f_\nu\right) + \frac{4}{3}\tan^2\frac{h}{4}(m_0 - m_\nu). \tag{43}$$

For the periodic spline, $f_0 = f_\nu$ and $m_0 = m_\nu$, therefore

$$\sigma_\nu(f) = \frac{4}{3}\,\frac{\sin\frac{3h}{4}}{\cos^3\frac{h}{4}}\sum_{i=0}^{\nu} f_i. \tag{44}$$

Using the relation (26) and the proposition 4.2, we obtain the second trigonometric spline quadrature formula

$$\tilde{\sigma}_\nu(f) = \int_a^b \tilde{s}_\Delta(x)\,dx =$$
$$= \sum_{i=0}^{\nu-1}\int_{x_i}^{x_{i+1}} \tilde{s}_\Delta(x)\,dx = \sum_{i=0}^{\nu-1}\int_{x_i}^{x_{i+1}} q_i(x)\,dx =$$
$$= \frac{2}{3}\sum_{i=0}^{\nu-1}\tan\frac{h_i}{4}\,\frac{13\cos^2\frac{h_i}{4} - 10}{4\cos^2\frac{h_i}{4} - 3}(f_i + f_{i+1}) - \tag{45}$$
$$-\frac{8}{3}\sum_{i=0}^{\nu-1}\frac{\sin^3\frac{h_i}{4}}{\cos\frac{3h_i}{4}}(M_i + M_{i+1}),$$

where $(M_i)_{i=0}^{\nu}$ satisfy the relation (25) and (39).

When the intervals are of equal length h, then

$$\tilde{\sigma}_\nu(f) = \frac{2}{3}\tan\frac{h}{4}\,\frac{13\cos^2\frac{h}{4} - 10}{4\cos^2\frac{h}{4} - 3}\left(f_0 + 2\sum_{i=1}^{\nu-1} f_i + f_\nu\right) - \tag{46}$$
$$-\frac{8}{3}\,\frac{\sin^3\frac{h}{4}}{\cos\frac{3h}{4}}\left(M_0 + 2\sum_{i=1}^{\nu-1} M_i + M_\nu\right).$$

For the periodic trigonometric spline functions of interpolation, $f_0 = f_\nu$ and $M_0 = M_\nu$, therefore

$$\tilde{\sigma}_\nu(f)=\frac{4}{3}\tan\frac{h}{4}\,\frac{13\cos^2\frac{h}{4}-10}{4\cos^2\frac{h}{4}-3}\sum_{i=0}^{\nu} f_i-\frac{16}{3}\,\frac{\sin^3\frac{h}{4}}{\cos\frac{3h}{4}}\sum_{i=0}^{\nu} M_i. \tag{47}$$

5. NUMERICAL EXAMPLES

Let $k\in N^*$, and I_k^H (resp. I_k^{QH}, $I_k^{S_1}$, $I_k^{S_2}$) be the numerical result obtained by usage of the trigonometric type Hermite's (resp. trigonometric quasi-Hermite, first kind trigonometric spline, second kind trigonometric spline) quadrature formula applied for computing the integral $I_k := \int_{a_k}^{b_k} f_k(x)\,\mathrm{d}x$.

Exemple 1. Let $I_1 := \int_0^2 e^{-x^2}\mathrm{d}x$ and $I_2 := \int_0^{2\pi}\frac{\sin^2 x}{5+4\cos x}\mathrm{d}x$. Using the above formulas and divisions of the integration interval with $\nu+1$ equidistant knots, we find the results presented in tables 1 and 2.

Table 1

ν	I_1^{QH}	I_2^{QH}	$\lvert I_1^{QH}-I_1^H\rvert$	$\lvert I_2^{QH}-I_2^H\rvert$
15	0.8820822469	0.8271403966	7.14E–7	4.18E–2
31	0.8820814442	0.7955839763	4.45E–8	1.02E–2
63	0.8820813941	0.7879272114	2.78E–9	2.53E–3
127	0.8820813909	0.7860293465	1.74E–10	6.31E–4
255	0.8820813908	0.7855558919	1.08E–11	1.58E–4
511	0.8820813908	0.7854375913	6.78E–13	3.94E–5

Table 2

ν	I_1^{S1}	I_2^{S1}	$\lvert I_1^{S1}-I_1^{S2}\rvert$	$\lvert I_2^{S1}-I_2^{S2}\rvert$
15	0.8791285161	0.7831303346	9.90E–7	2.99E–4
31	0.8807826734	0.7851177017	3.19E–8	9.83E–6
63	0.8814719213	0.7853630863	9.99E–10	3.11E–7
127	0.8817861009	0.7853937799	3.12E–11	9.74E–9
255	0.8819360435	0.7853976156	9.73E–13	3.05E–10
511	0.8820092838	0.7853980949	3.04E–14	9.52E–12

Example 2. Let $I_3 := \int_0^{2\pi}\frac{1}{\sqrt{2}+\cos x}\mathrm{d}x$ and $I_4 := \int_0^{2\pi}\frac{1}{5+4\cos x}\mathrm{d}x$. Working with nonequidistants knots, a better approximation of the function under integral sign is obtained, therefore a better aproximation of the definite integral is

possible. We illustrate this aspect in the following. Let us consider $\Delta_1 := \{0, 0.1, 0.2, 0.28, 0.34, 0.38, 0.4, 1.4(0.2), 6.2(0.1), 6.25, 6.27, 6.28, 6.283185307\}$ and $\Delta_2 := \{0, 0.2(0.05), 0.4(0.02), 1(0.1), 6.25(0.05), 6.27, 6.28, 6.283185307\}$, two nonequidistant divisions of the interval $[0, 2\pi]$. Using the quadrature formulas (42) and (45) we obtain the numerical results from table 3.

Table 3

Division	I_3^{S1}	I_4^{S1}	$\lvert I_3^{S1}-I_3^{S2}\rvert$	$\lvert I_4^{S1}-I_4^{S2}\rvert$
Δ_1	6.2831808026	2.0943935193	6.91E–6	2.98E–6
Δ_2	6.2831851630	2.0943950557	1.74E–7	6.55E–8

All numerical results were obtained using a Turbo Pascal program, working in extended floating point precision (the Turbo Pascal type **extended**), on MSDOS 6.0 personal computer.

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