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AN ALGORITHM FOR MULTICRITERIA TRANSPORTATION PROBLEMS

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1. PRELIMINARIES

It is known that many applications of linear programming in managing economic processes come to the solving some transportation problems. A such model was presented for instance in [8].

A transportation problem (of the cost type) is a linear programming problem of the following type

(C) $$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ i \in \{1, \ldots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ j \in \{1, \ldots, n\}$$

$$x_{ij} \geq 0, (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\},$$

as usually represented in a table of the following type

c_{11}	…	c_{1n}	a_1
…	…	…	…
c_{m1}	…	c_{mn}	a_m
b_1	…	b_n	

Any feasible solution of the problem (C) is called a transport plan. A chain is any system of cells of the type

$$(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots$$

or

$$(i_1, j_1), (i_2, j_1), (i_2, j_2), (i_3, j_2), \ldots$$

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such that any pair of two adjacent cells are situated either in the same row or in the same column and any system formed by three cells from the chain is not situated in the same row or in the same column. If the last cell of the chain is in the same row or column with the first cell, the chain is called a cycle.

A transport plan $X = (x_{ij})$ is acyclic if the cells that correspond to $x_{ij} > 0$ do not contain any cycle.

It is known that, if a transportation problem admits a transport plan, then it admits a least an cyclic transport plan.

If, in an acyclic transport plan $X = (x_{ij})$, the number of elements $x_{ij} > 0$ is $m + n - 1$, then the plan is called nondegenerated. If this number is smaller than $m + n - 1$, then the transport plan is called degenerated. If the transport plan $X = (x_{ij})$ is degenerated, it will be convenient to add to the set

$$\{(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}: x_{ij} > 0\},$$

some elements $(h, k) \in \{1, \dots, m\} \times \{1, \dots, n\}$ such that the new set has $m + n - 1$ elements and the cells that correspond to it do not form a cycle; such a set is called a selection set generated by X. Obviously, in general, we can generate more selection sets. The family of the selection sets generated by the plan X will be denoted by Sel(X).

The acyclic transport plan $X = (x_{ij})$ is called potential with respect to a selection set $A \in$ Sel(X), if there exist the real numbers

$$u_1, \dots, u_m, v_1, \dots, v_n$$

which satisfy the conditions

(1) $$v_j - u_i \le c_{ij} \text{ for all } (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$$

and

(2) $$v_j - u_i = c_{ij} \text{ for all } (i, j) \in A.$$

If the real numbers $u_1, \dots, u_m, v_1, \dots, v_n$ satisfy (1)–(2), then the $(m + n)$-tuple $(u_1, \dots, u_m, v_1, \dots, v_n)$ is called a potential system for A.

THEOREM 1.1. (see, for example [11]). *The transport plan X is an optimal plan if and only if there is a set $A \in$ Sel(X) such that X is a potential plan with respect to A.*

2. MULTICRITERIA TRANSPORTATION PROBLEMS

In the following we call a multicriteria transportation problem of the cost type, denoted by (MTP), a multicriteria linear programming problem in which the objective function is a vector function $f = (f_1, \dots, f_p) : \mathbb{R}^{m \times n} \to \mathbb{R}^p$, given by

$$f_k(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij} \quad k \in \{1, \dots, p\},$$

for all $X = (x_{ij}) \in \mathbb{R}^{m \times n}$, and the constraints are

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i, \ i \in \{1, \dots, m\} \\ \sum_{i=1}^{m} x_{ij} = b_j, \ j \in \{1, \dots, n\} \\ x_{ij} \ge 0, \ (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\} \end{cases}$$

By analogy with the scalar case, a multicriteria transportation problem (MTP) will be represented by

$c_{11}^1, \dots, c_{11}^p$	...	$c_{1n}^1, \dots, c_{1n}^p$	a_1
...	...	...	...
$c_{m1}^1, \dots, c_{m1}^p$	...	$c_{mn}^1, \dots, c_{mn}^p$	a_m
b_1	...	b_n	

The set of the feasible solution of the problem (MTP) will be denoted by S. A transport plan $X \in S$ is called Pareto (or min-efficient) if there is no $Y \in S$ such that

$$f_k(Y) \le f_k(X), \quad k \in \{1, \dots, p\}.$$

at least one of the inequalities being strict. Because any Pareto transport plan is a Pareto solution of a multicriteria linear programming problem, some interesting properties of the Pareto transport plan set can be found for instance in [7]. On the other hands, for the determination of a Pareto transport plan, we can use any algorithms given in [2], [9], [10] etc. If, in addition, x_{ij} ($(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$) must be integer, the algorithm given in [7] allows us to determine all the equivalence classes of the Pareto transport plans. Note that we can also solve a multicriteria transportation problem using the r-balance points [6].

The particular form of the multicriteria linear programming problem which corresponds to the multicriteria transportation problem (MTP) allows us to elaborate specific algorithms, as we can see below.

In the following we will denote by (T_k) ($k \in \{1, \dots, p\}$) the scalar transportation problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0, \ (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}.$$

Let $X = (x_{ij})$ be an acyclic transport plan and let $A \in \mathrm{Sel}(X)$. For each $k \in \{1, \dots, p\}$, let $(u_1^k, \dots, u_m^k, v_1^k, \dots, v_n^k)$ be a solution of the system

$$v_j^k - u_i^k = c_{ij}^k, \ (i, j) \in A.$$

We denote by

$$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k$$

for each $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ and $k \in \{1, \dots, p\}$. The following result gives a sufficient condition for the Pareto-efficiency:

THEOREM 2.1. *Let $k \in \{1, \dots, p\}$. If X is a potential plan of the problem (T_k), with respect to $A \in \mathrm{Sel}(X)$ and if*

(3) $$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k < 0$$

for each

$$(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\} \setminus A$$

where

$$(u_1^k, \dots, u_m^k, v_1^k, \dots, v_n^k)$$

is a solution of the system

$$v_j^k - u_i^k = c_{ij}^k, \quad (i, j) \in A$$

then X is a Pareto transport plan for the multicriteria transportation problem (MTP).

Proof. If X is a potential plan for the problem (T_k), then it is an optimal plan for the problem (T_k). From (3), it follows that X is the unique optimal transport plan for (T_k). Hence X is a Pareto transport plan for the multicriteria transportation problem (MTP). □

Let $X = (x_{ij})$ be a potential plan for the problem (T_1) and let $A \in \mathrm{Sel}(X)$. For each $k \in \{1, \dots, p\}$ let $(u_1^k, \dots, u_m^k, v_1^k, \dots, v_n^k)$ be a solution of the system

(4) $$v_j^k - u_i^k = c_{ij}^k, \quad (i, j) \in A$$

We denote by

(5) $$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k$$

for each $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ and $k \in \{1, \dots, p\}$. Obviously $\alpha_{ij}^1 < 0$, for all $(i, j) \in A_X^0 := \{1, \dots, m\} \times \{1, \dots, n\}$. Let

$$A_X^1 = \{(i, j) \in A_X^0 : \alpha_{ij}^1 = 0\}.$$

We presume that $\alpha_{ij}^2 \leq 0$, for all $(i, j) \in A_X^1$, and we denote by

$$A_X^2 = \{(i, j) \in A_X^1 : \alpha_{ij}^2 = 0\}.$$

We continue in the same way: if there is $k \in \{1, \dots, p-1\}$, for which $\alpha_{ij}^k \leq 0$ for all $(i, j) \in A_X^{k-1}$, we denote by

$$A_X^k = \{(i, j) \in A_X^{k-1} : \alpha_{ij}^k = 0\}.$$

A necessary condition for the Pareto-efficiency is given by the following result.

THEOREM 2.2. *If there exists $q \in \{1, \dots, p-1\}$ such that*

$$\alpha_{ij}^k \leq 0, \textit{ for all } (i, j) \in A_X^{k-1}, \textit{ and } k \in \{1, \dots, q\},$$

and if there exists $(r, s) \in A_X^q = \{(i, j) \in A_X^{q-1} : \alpha_{ij}^q = 0\}$, with $\alpha_{rs}^{q+1} > 0$, then there is a transport plan Y having the property that

$$f_k(Y) = f_k(X) \textit{ for each } k \in \{1, \dots, q\}$$

and

$$f_{q+1}(Y) < f_{q+1}(X),$$

which means that X is not Pareto-efficient.

Proof. We introduce the cell (r, s) in the transport plan $X = (x_{ij})$. This will have a cycle $\mathscr{C}$. We travel through this cycle, starting from the cell (r, s) and we assign to its cells the sign + and –, alternatively, starting with the cell (r, s) which gets the + sign. The cells of the cycle denoted by + form a semichain L^+, and the cells of the cycle denoted by – form a semichain L^-. We analyse the elements x_{ij} of the transport plan X situated in the semichain L^- and we define

$$\theta = \min\{x_{ij} : (i, j) \in L^-\},$$

which is attained, for example, in the cell (u, t). From the elements x_{ij} which correspond to the semichain L^- we substract the number θ, and to the elements x_{ij} corresponding to the semichain L^+ we add the number θ. The other elements, which are not in the cycle $\mathscr{C}$, remain the same. We obtain a new transport plan Y to which we attach the set

(6) $$B = A \cup \{(r,s)\} \setminus \{(u,t)\}.$$

Let us denote by M the set of the cells which are not in the cycle $\mathscr{C}$. Then, for each $k \in \{1, \ldots, q\}$, we have

$$f_k(Y) = \sum_{(i,j)\in M} c_{ij}^k y_{ij} + \sum_{(i,j)\in L^+} c_{ij}^k y_{ij} + \sum_{(i,j)\in L^-} c_{ij}^k y_{ij} =$$
$$= \sum_{(i,j)\in M} c_{ij}^k x_{ij} + \sum_{(i,j)\in L^+} c_{ij}^k (x_{ij} + \theta) + \sum_{(i,j)\in L^-} c_{ij}^k (x_{ij} - \theta) =$$
$$= f_k(X) - \theta\alpha_{rs}^k = f_k(X).$$

Calculating $f_{q+1}(Y)$, we obtain

$$f_{q+1}(Y) = \sum_{(i,j)\in M} c_{ij}^{q+1} y_{ij} + \sum_{(i,j)\in L^+} c_{ij}^{q+1} y_{ij} + \sum_{(i,j)\in L^-} c_{ij}^{q+1} y_{ij} =$$
$$= \sum_{(i,j)\in M} c_{ij}^{q+1} x_{ij} + \sum_{(i,j)\in L^+} c_{ij}^{q+1} (x_{ij} + \theta) + \sum_{(i,j)\in L^-} c_{ij}^{q+1} (x_{ij} - \theta) =$$
$$= f_{q+1}(X) - \theta\alpha_{rs}^{q+1} < f_{q+1}(X).$$

Hence, the transport plan X is not a Pareto one. □

3. THE ALGORITHM

Using theorems 2.1 and 2.2, we can state the following algorithm for the determination of a Pareto transport plan for multicriteria transportation problems.

Algorithm

1. We put $k = 1$ and $A_X^0 = \{1, \ldots, m\} \times \{1, \ldots, n\}$.
2. One determines an cyclic optimal transport plan $X = (x_{ij})$ for the problem (T_1) and we attach to it a set $A \in \mathrm{Sel}(X)$.
3. We determine a potential system
$$(u_1^k, \ldots, u_m^k, v_1^k, \ldots, v_n^k)$$
given by

(7) $$v_j - u_i = c_{ij}^k, \quad (i,j) \in A.$$

4. We consider
$$\alpha_{ij}^k = v_j^k - u_i^k - c_{ij}^k$$
for each $(i,j) \in A_X^{k-1}$.
5. We compare to zero each of the numbers
$$\alpha_{ij}^k, \quad (i,j) \in A_X^{k-1} \setminus A.$$

a) If $\alpha_{ij}^k < 0$, for each $(i,j) \in A_X^{k-1} \setminus A$, then X is a Pareto transport plan for the multicriteria transportation problem (MTP) and the algorithm stops.

b) If there is $(r,s) \in A_X^{k-1} \setminus A$ so that $\alpha_{rs}^k > 0$, then we go to step 10.

c) If there is $(r,s) \in A_X^{k-1} \setminus A$ such that $\alpha_{rs}^k = 0$ and $\alpha_{ij}^k > 0$, for each $(i,j) \in A_X^{k-1} \setminus A$, then we go to step 6.

6. We compare k to p.

a) If $k = p$, then X is a Pareto transport plan for the multicriteria transportation problem (MTP) and the algorithm stops.

b) If $k < p$, then we go to step 7.

7. We put $A_X^k = \{(i,j) \in A_X^{k-1} : \alpha_{ij}^k = 0\}$.
8. We compare $A_X^k \setminus A$ to the empty set.

a) If $A_X^k \setminus A = \varnothing$, then X is a Pareto transport plan for the multicriteria transportation problem (MTP) and the algorithm stops.

b) If $A_X^k \setminus A \neq \varnothing$, then we go to step 9.

9. k increase with 1 and we go back to step 3.
10. We introduce the cell (r, s) in the transport plan X. This will have a cycle $\mathscr{C}$. We travel through this cycle, starting from the cell (r, s) and we assign to its cells by the sign + and −, alternatively, starting with the cell (r, s) which gets the + sign. The cells of the cycle denoted by + form a semichain L^+, and the cells of the cycle denoted by − form a semichain L^-. We analyse the elements x_{ij} of the transport plan X situated in the semichain L^- and we define
$$\theta = \min\{x_{ij} : (i,j) \in L^-\},$$
which is attained, for example, in the cell (u, t). From the elements x_{ij} which correspond to the semichain L^- we substract the number θ, and to the elements x_{ij} corresponding to the semichain L^+ we add the number θ. The other elements, which are not in the cycle $\mathscr{C}$, remain the same. We obtain a new transport plan X to which we attach the set
$$A := A \cup \{(r,s)\} \setminus \{(u,t)\}.$$
We go back to step 3.

In order to illustrate the above algorithm we conclude with the following numerical example.

Example. Let us consider the (MTP) problem given by Table 1.

In this case we have $p = 3$, $m = 3$, $n = 4$. We put $k = 1$ and $A_X^0 = \{1, 2, 3\} \times \{1, 2, 3, 4\}$. An acyclic optimal transport plan X for the problem (T_1) is

Table 1

4	3	5	3	5	3	6	8	1	6	8	4	102
2	4	2	1	2	2	8	4	1	4	5	8	136
4	2	1	3	4	3	5	6	7	9	8	1	172
151			122			83			54			

$$X = \begin{pmatrix} 48 & 0 & 0 & 54 \\ 103 & 33 & 0 & 0 \\ 0 & 89 & 83 & 0 \end{pmatrix}$$

We have

$$A = \{(1, 1), (1, 4), (2, 1), (2, 2), (3, 2), (3, 3)\}.$$

A solution of the system (7) is

$$(u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, v_3^1, v_4^1) = (0, 2, 0, 4, 3, 5, 7).$$

Since there is $(1, 2) \in A_X^0 \setminus A$ such that $\alpha_{12}^1 = 0$ and $\alpha_{ij}^1 \leq 0$ for each $(i, j) \in A_X^0 \setminus A$ and $k < p$, we go to step 7. We have

$$A_X^1 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2)\}.$$

Since $A_X^1 \setminus A \neq \varnothing$ we put $k = 2$ and we go to step 3. A solution of the system (7) is

$$(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, -1, -3, 3, 1, 3, 8).$$

Since there is $(2, 4) \in A_X^1 \setminus A$ such that $\alpha_{24}^2 = 4 > 0$, we go to step 10. We have $L^+ = \{(1, 1), (2, 4)\}$ and $L^- = \{(2, 1), (1, 4)\}$, $\theta = 54$ and

$$A = \{(1, 1), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3)\}.$$

A solution of the system (7) is

$$(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, -1, -3, 3, 1, 3, 4).$$

Since there is $(3, 1) \in A_X^1 \setminus A$ so that $\alpha_{31}^2 = 4 > 0$, we go to step 10. We have $L^+ = \{(2, 2), (3, 1)\}$, $L^- = \{(2, 1), (3, 2)\}$ and $\theta = 49$. The new A is

$$A = \{(1, 1), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3)\}.$$

A solution of the system (7) is

$$(u_1^2, u_2^2, u_3^2, v_1^2, v_2^2, v_3^2, v_4^2) = (0, 3, 1, 3, 5, 7, 8).$$

Since there is $(1, 2) \in A_X^1 \setminus A$ so that $\alpha_{12}^2 = 0$ and $\alpha_{ij}^2 \leq 0$ for each $(i, j) \in A_X^1 \setminus A$ and $k < p$, we go to step 7. We have

$$A_X^2 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3)\}.$$

Since $A_X^2 \setminus A \neq \varnothing$ we put $k = 3$ and go to step 3. A solution of the system (7) is

$$(u_1^3, u_2^3, u_3^3, v_1^3, v_2^3, v_3^3, v_4^3) = (0, 5, 4, 5, 7, 11, 13).$$

There is $(1, 4) \in A_X^2 \setminus A$ such that $\alpha_{14}^3 = 9 > 0$. Then we go to step 10. We have

$$L^+ = \{(1, 4), (2, 2), (3, 1)\}, \; L^- = \{(1, 1), (2, 4), (3, 2)\},$$

$\theta = 40$,

$$A = \{(1, 1), (1, 4), (2, 2), (2, 4), (3, 1), (3, 3)\}.$$

and

$$X = \begin{pmatrix} 62 & 0 & 0 & 40 \\ 0 & 120 & 0 & 14 \\ 89 & 0 & 123 & 0 \end{pmatrix} \tag{8}$$

A solution of the system (7) is

$$(u_1^3, u_2^3, u_3^3, v_1^3, v_2^3, v_3^3, v_4^3) = (0, -4, 4, 5, -2, 11, 4).$$

Since $\alpha_{ij}^3 < 0$, for all $(i, j) \in A_X^2 \setminus A$, we have that X given by (8) is a Pareto transport plan for the multicriteria transportation problem given by Table 1.

REFERENCES

[1] R. Avram-Niţchi, *About the Multiple Objective Fuzzy Operatorial Transportation Problems*, Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate. Cluj-Napoca, 21–26 mai, 1997, 1–4.

[2] A. Baciu, A. Pascu, E. Puşcaş, *Applications of the Operations Research* (in Roumanian), Bucureşti, Ed. Militară, 1988.

[3] A. Gupta, S. Khanna, M. C. Puri, Paradoxical Situations in Transportation Problems. Cahiers du C.E.R.O., **34**, *1* (1992), 37–49.

[4] D. S. Hochbaum, S. P. Hong, On the Complexity of the Production-Transportation Problem, *SIAM J. Optimization*, **6**, *1* (1996), 250–264.

[5] L. Lupşa, E. Duca, D. I. Duca, On the Structure of the Set of Points Dominated and Nondominated in an Optimization Problem. *Revue d'anal. num. et la théorie de l'approximation*, **22**, *2* (1993), 193–199.

[6] L. Lupşa, D. I. Duca, E. Duca, On the Balanced and Nonbalanced Vector Optimization Problems, *Revue d'anal. num. et la théorie de l'approximation*. **24**, *1* (1995), 112–124.

[7] L. Lupşa. D. I. Duca, E. Duca, Equivalence Classes in the Set of Efficient Solutions, *Revue d'anal. num. et la théorie de l'approximation*, **25**, *1–2* (1996), 127–136.

[8] L. Lupşa, D. I. Duca, E. Duca, On Bicriterial Transportation Problems, *Revue d'anal. num. et la théorie de l'approximation*, **27**, *1* (1998), 81–90.

[9] V. V. Podinovsky, V. D. Noghin, *Pareto-Optimal Solutions of Multicriteria Problems* (in Russian), Moscow, Nauka, 1982.

[10] M. Zeleny, *Multiple Criteria Decision Making*, New York, McGraw-Hill 1982.

[11] S. I. Zuhovicki, L. I. Avdeeva, *Linear and Convex Programming* (in Russian), Moscow, Nauka 1964.

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