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A NEW PROOF FOR THE APPROXIMATION OF THE LOG-FUNCTION BY KANTOROVICH POLYNOMIALS IN THE L_p-NORM

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Twenty years ago the author solved the saturation problem for the Kantorovich Polynomials in the L_1-*norm* (cf. [2]). The most difficult part of this saturation problem had been the proof of the direct theorem and herein especially the estimate for the approximation of the *Log*-function. In the meantime this result was often used in other papers.

We are now able to give a new and essentially shorter proof with a smaller constant of the estimate

$$\|P_n \log - \log\|_1 = O\left((n+1)^{-1}\right).$$

The approximation in the L_p-*norm* for $p > 1$ is due to Riemenschneider [3], who therein uses an inequality of the proof for the L_1-*norm* [1]. Now we are also able to give a very short form for Riemenschneider's proof.

For an $f \in L_1(I^*)$, $I^* = (0, 1)$, the *nth* Kantorovich Polynomial on I^* is defined by

$$P_n(f, x) = \int_0^1 K_n(x, t) f(t) \mathrm{d}t$$

with the kernel $K_n(\cdot, \cdot)$ given by

$$K_n(x, t) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\chi_{I_k}(t),$$

where χ_{I_k} denotes the characteristic function on $I_k := \left(\frac{k}{n+1}, \frac{k+1}{n+1}\right)$ and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Let $F(x) = \int_0^x f(t)dt$, then there is a relation between the Kantorovich and the

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well known Bernstein polynomials $B_n(F,x)=\sum_{k=0}^{n} p_{n,k}(x)F\left(\frac{k}{n}\right)$, namely

$$\frac{\mathrm{d}}{\mathrm{d}x}B_{n+1}(F,x)=P_n(f,x). \tag{1}$$

For all properties of the Bernstein and the Kantorovich Polynomials which we will mention here see e.g. chapter 10 of the book of De Vore and Lorentz [1]. For abbreviation we will not refer to the original papers.

We will now proof the following theorem.

THEOREM 1.

$$\|P_n\log-\log\|_1=O\left((n+1)^{-1}\right).$$

Proof. A short computation gives for the error function E_n

$$E_n(x)=P_n(\log,x)-\log x=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{I_k}\log u\,\mathrm{d}u-\log x=$$

$$=\sum_{k=0}^{n}p_{n,k}(x)\left[\log\frac{a_k}{(n+1)}-1\right]-\log x,$$

where $a_0=1$ and $a_k=\dfrac{(k+1)^{k+1}}{k^k}$.

Differentiating E_n and using $p'_{n,k}(x)=n(p_{n-1,k-1}(x)-p_{n-1,k}(x))$ with $p_{n,-1}(x)=0$ gives

$$E'_n(x)=n\sum_{k=0}^{n-1}p_{n-1,k}(x)\log\frac{a_{k+1}}{a_k}-\frac{1}{k}=$$

$$=\frac{1}{x}\left(\sum_{k=0}^{n-1}p_{n,k+1}(x)(k+1)\log\frac{a_{k+1}}{a_k}-1\right)=$$

$$=\frac{(1-x)^n}{x}\left(\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{-k}\left(\log\left(\frac{a_k}{a_{k-1}}\right)^k-1\right)-1\right)=$$

$$=\frac{(1-x)^n}{x}g_{n(x)},$$

where $g_n(x)=\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{-k}(\log d_k-1)-1$ and $d_k=\left(\dfrac{a_k}{a_{k-1}}\right)^k$.

We differentiate g_n and obtain $g'(x)>0$ for all $x\in I^*$ if $\log d_k-1\geq 0$ for all $k\in\mathbb{N}$.

For the last assumption let us denote

$$b_k=\left(\frac{k+1}{k}\right)^{k+1/2},\quad k\in\mathbb{N}.$$

It is known that

$$b_k>b_{k+1}>e,\quad k\in\mathbb{N}.$$

Then we can write $d_k=b_k\dfrac{b_{k-1}}{b_{k^2-1}}$, for $k=2,3,\ldots$ ($d_1=4$) and this implies $d_k>b_k>e$ and therefore $\log d_k-1>0$ for all $k\in\mathbb{N}$.

Now we have $g'_n(x)>0$ for all $x\in I^*$ and therefore g_n is monotone increasing in I^*. Moreover we can easily see, that g_n changes sign from minus to plus in I^*. Thus it finally follows that g_n has exactly one zero in I^*. Then E'_n has exactly one zero in I^*. Moreover we can show that E_n changes sign from plus to minus in I^* and because E'_n has exactly one zero in I^* we get that E_n has exactly one zero x_s, $x_s=x_s(n)$ in I^*.

So we can compute

$$\|P_n\log-\log\|_1=\int_0^1|P_n(\log,x)-\log x|\mathrm{d}x=$$

$$=\int_0^{x_s}(P_n(\log,x)-\log x)\mathrm{d}x+\int_{x_s}^{1}(\log x-P_n(\log,x))\mathrm{d}x.$$

Now we write

$$H(x)=\begin{cases}\int_0^x\log t\,\mathrm{d}t=x\log x-x, & x\in(0,1]\\ 0, & x=0\end{cases}$$

Since H is convex on $I=[0,1]$ it follows that $B_{n+1}(H,\cdot)$ is convex on I, too. Moreover $B_{n+1}(H,0)=H(0)$ and $B_{n+1}(H,1)=H(1)$. Thus we get by (1)

$$\|P_n\log-\log\|_1=2\left(B_{n+1}(H,x_s)-H(x_s)\right).$$

To give an estimation of the last term let us look at f a convex function on I. Then for each system of points $x_1,x_2,\ldots,x_{n+1}$ from I and for non negative numbers $c_1,c_2,\ldots,c_{n+1}$ with $\sum_{k=1}^{n+1}c_k=1$ one has

$$f\left(\sum_{k=1}^{n+1}c_kx_k\right)\leq\sum_{k=1}^{n+1}c_kf(x_k).$$

Choosing $c_k=\dfrac{1}{x}\dfrac{k}{n+1}\dbinom{n+1}{k}x^k(1-x)^{n-k+1}$, $x_k=\dfrac{k}{n+1}$ and $f(x)=-\log x$, we obtain

$$B_{n+1}(H;x) \le x\log\left(x+\frac{1-x}{n+1}\right).$$

Taking into account that $\ln b - \ln a < \frac{b-a}{\sqrt{ab}}$, $0<a<b$, and $x_s \in I^*$ we conclude with

$$B_{n+1}(H,x_s) - H(x_s) \le \frac{1-x_s}{n+1} \le \frac{1}{n+1}. \ \square$$

For the saturation problem in case $p>1$ Riemenschneider proved the direct theorem, see [3]. He could reduce the problem to the estimation of $\|x(P_n(\log,x)-\log x\|_\infty = O\left((n+1)^{-1}\right)$ and he did this with an inequality for the integrand given in the first L_1-norm proof in [2]. We can now prove the following theorem in a different way, too.

THEOREM 2.

$$\|x(P_n(\log,x)-\log x\|_\infty = O\left((n+1)^{-1}\right).$$

Proof. We write

$$x(P_n(\log t,x)-\log x) = \left[xP_n(\log t,x) - B_{n+1}(t\log t,x)\right] + \left[B_{n+1}(t\log t,x) - x\log x\right].$$

In order to estimate the norm of the right hand side we first look at the second term and define $G(x) = x\log x$. G has the same properties as H in the proof of theorem 1 and so the sup-norm of the second term can be estimated by $O\left((n+1)^{-1}\right)$. So it only remains to estimate the norm of the first term.

Applying $\lim_{k\to 0}\frac{k}{(n+1)}\cdot\log\frac{k}{n+1}=0$ and $\binom{n+1}{k}\cdot\frac{k}{n+1}=\binom{n}{k-1}$, we obtain

$$\left|xP_n(\log t,x) - B_{n+1}(t\log t,x)\right| = \left|\sum_{k=1}^{n} xp_{n,k}(x)\left(\log\left(\frac{k+1}{k}\right)^k - 1\right) - x(1-x)^n\right|.$$

For $0<k$ we have $\left|\log\left(\frac{k+1}{k}\right)^k - 1\right| \le \frac{1}{2k}$ and thus

$$\left|xP_n(\log t,x) - B_{n+1}(t\log t,x)\right| \le x(1-x)^n + \sum_{k=1}^{n} xp_{n,k}(x)\frac{1}{2k} \le$$

$$\le \sum_{k=0}^{n} xp_{n,k}(x)\frac{1}{k+1} \le \frac{1}{n+1}\sum_{k=-1}^{n} p_{n+1,k+1}(x) = \frac{1}{n+1}.$$

This finishes the proof. $\square$

REFERENCES

1. R. A. De Vore and G. G. Lorentz. *Constructive Approximation*, Springer-Verlag, 1993.
2. V. Maier, The L_1 saturation class of the Kantorovich operator. *J. Approx. Theory*, **22** (1978), 227–232.
3. S. D. Riemenschneider, The L_p-saturation of the Kantorovich-Bernstein polynomials. *J. Approx. Theory*, **23** (1978), 158–162.

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