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# ON A PROBLEM OF B. A. KARPILOVSKAJA

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In [7] one consider the following problem:

(1) $$y^{(2p)}(t)-\varphi_1(t)\,y^{(2p-1)}(t)-\cdots-\varphi_{2p}(t)\,y(t)=f(t),\quad t\in[a,b]$$

(2) $$y^{(q)}(a)=y^{(q)}(b)=0,\quad q=0,1,2,\dots,p-1,$$

where $p\in\mathbb{N}$, $p\geq 1$. In the same paper one determines an approximate solution of the form

(3) $$\overline{y}(t)=(t-a)^p(t-b)^p\cdot\sum_{k=1}^{n}c_k t^{k-1},\ t\in[a,b]$$

where the coefficients $c_k$, $k=1,2,\dots,n$ are determined from the system of equations:

(4) $$\overline{y}^{(2p)}(t_i)-\varphi_1(t_i)\,\overline{y}^{(2p-1)}(t_i)-\cdots-\varphi_{2p}(t_i)\,\overline{y}(t_i)=f(t_i),\quad i=1,2,\dots,n,$$

where $t_i$, $i=\overline{1,n}$ are the nodes of a partition

(5) $$\Delta_n':=a<t_1<t_2<\cdots<t_n<b$$

of the interval $[a,b]$.

In the case when the nodes of the partition $\Delta_n'$ are the roots of the Chebyshev polynomial it is given an upper delimitation of the norm $\|y-\overline{y}\|_\infty$, where $y$ is the exact solution of the problem (1)–(2). From this delimitation it follows that the order of approximation of the exact solution by the functions $\overline{y}$ given by (3) is $O\left(\frac{\ln n}{n}\right)$.

In the following, taking as an approximant of the exact solution of the problem (1)–(2) a spline function belonging to the space $S_{2m+2p-1}(\Delta_n)$ of $2p$-derivative-interpolating spline functions, defined in [9], one proves that the

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order of approximation is at least $O\left(\frac{1}{n\sqrt{n}}\right)$.

DEFINITION 1. *Let* $m, n, p \in \mathbb{N}$, $n \geq 2, p \geq 1, m \geq 2, m+p \leq n+1$ *and let*

$$\Delta_n := -\infty = t_{-1} < a = t_0 < t_1 < \cdots < t_n = b < t_{n+1} = +\infty$$

*be a partition of the interval* $[a, b]$.

*A function* $s : \mathbb{R} \to \mathbb{R}$ *satisfying the conditions*

$1^0$ $s \in C^{2m+p-2}(\mathbb{R})$;

$2^0$ $s|_{I_k} \in \mathscr{P}_{2m+p-1}$, $I_k = [t_{k-1}, t_k)$, $k = 1, 2, \ldots, n$;

$3^0$ $s|_{I_0} \in \mathscr{P}_{m+p-1}$, $I_0 = [t_{-1}, t_l)$, $I_{n+1} = [t_n, t_{n+1})$,

*is called a natural spline function of degree* $2m + p - 1$.

Here $\mathscr{P}_r$ $(r \in \mathbb{N})$ stands for the set of polynomials of degree at most $r$.

Denoting by $S_{2m+p-1}(\Delta_n)$ the set of all functions verifying the conditions $1^0$–$3^0$ from Definition 1, one sees that each $s \in S_{2m+p-1}(\Delta_n)$ admits a representation of the form

$$s(t) = \sum_{i=0}^{m+p-1} A_i t^i + \sum_{k=0}^{n} a_k (t - t_k)_+^{2m+p-1}, \quad t \in \mathbb{R} \tag{6}$$

where

$$\sum_{k=0}^{n} a_k t_k^j = 0, \quad j = 0, 1, 2, \ldots, m-1 \tag{7}$$

and

$$(t - t_k)_+ = \begin{cases} 0, & \text{if } t \leq t_k, \\ t - t_k, & \text{if } t > t_k \end{cases} \quad t \in [a, b]. \tag{8}$$

(see Theorem 2 from [9]).

Taking into account the representation (7) and the conditions (8), it follows that each $s \in S_{2m+p-1}(\Delta_n)$ depends on $n + p + 1$ free parameteres, so that $S_{2m+p-1}(\Delta_n)$ is a vector space of dimension $n + p + 1$ with respect to the usual (pointwise) of addition and multiplication by scalar of real functions.

The following theorem will allow us to use a spline function from $S_{2m+2p-1}(\Delta_n)$ as an approximant for the solution of the problem (1)–(2).

THEOREM 2. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *verifies the conditions:*

$$\begin{aligned} &f^{(q)}(a) = \alpha^{(q)}, \quad q = 0, 1, 2, \ldots, p-1 \\ &f^{(q)}(b) = \beta^{(q)}, \quad q = 0, 1, 2, \ldots, p-1 \\ &f^{(2p)}(t_k) = \gamma_k, \quad k = 0, 1, 2, \ldots, n \end{aligned} \tag{9}$$



*where* $t_k$, $k = \overline{0, n}$ *are the nodes of the partition* $\Delta_n$ *and* $\alpha^{(q)}$, $\beta^{(q)}$, $q = \overline{0, p-1}$ *and* $\gamma_k$, $k = \overline{0, n}$, *are given numbers.*

*Then there exists a unique spline function* $s \in S_{2m+2p-1}(\Delta_n)$ *such that*

$$\begin{aligned} &s_f^{(q)}(a) = \alpha^{(q)}, \quad q = \overline{0, p-1}, \\ &s_f^{(q)}(b) = \beta^{(q)}, \quad q = \overline{0, p-1}, \\ &s_f^{(2p)}(t_k) = \gamma_k, \quad k = \overline{0, n}. \end{aligned} \tag{10}$$

*Proof.* If $s_f$ is of the form (6), fulfilling the conditions (7), then, imposing the conditions (10), we find the system:

$$\begin{aligned} &\sum_{i=0}^{m+2p-q-1} \frac{(q+1)!}{i!} A_{q+i} t_0^i = \alpha^{(q)}, \quad q = \overline{0, p-1} \\ &\sum_{i=0}^{m+2p-q-1} \frac{(q+1)!}{i!} A_{q+i} t_n^i + \sum_{k=0}^{n} \frac{(2m+2p-1)!}{(2m+2p-q-1)!} a_k (t_n - t_k)^{2m+2p-q-1} = \beta^{(q)}, \\ &\qquad q = \overline{0, p-1} \\ &\sum_{i=0}^{m-1} \frac{(2p+1)!}{i!} A_{2p+i} t_j^i = \sum_{k=0}^{n} \frac{(2m+2p-1)!}{(2m-1)!} a_k (t_j - t_k)^{2m-1} = \gamma_j, \quad j = \overline{0, n} \\ &\sum_{k=0}^{n} a_k t_k^i = 0, \quad i = \overline{0, m-1} \end{aligned} \tag{11}$$

having $2p + n + 1 + m$ equations and the same number of unknowns: $A_0, A_1, \ldots A_{m+2p-1}, a_0, a_1, \ldots, a_n$.

This system has a unique solution if and only if the associated homogeneous system (obtained for $\alpha^{(q)} = 0 = \beta^{(q)}$, $q = \overline{0, p-1}$, $\gamma_k = 0$, $k = \overline{0, n}$) has only the null solution.

Let's show that, if $s \in S_{2m+2p-1}$ verifies $s^{(q)}(a) = s^{(q)}(b) = 0$, $q = \overline{0, p-1}$; $s^{(2p)}(t_k) = 0$, $k = \overline{0, n}$ then $s \equiv 0$ or $\mathbb{R}$.

Integrating by parts we obtain

$$\int_{t_0}^{t_n} \left[s^{(m+2p)}(t)\right]^2 dt = \sum_{j=0}^{m-2} (-1)^j s^{(m+2p+j)}(t) \cdot s^{(m+2p-j-1)}(t)\Big|_{t_0}^{t_n} + $$
$$+ (-1)^{m-1} \int_{t_0}^{t_n} s^{(2m+2p-1)}(t) \cdot s^{(2p+1)}(t)\, dt.$$

But $s^{(m+2p+j)}(t_0)=s^{(m+2p+j)}(t_n)=0,\ j=\overline{0,m-2}$ (by Condition $3^0$ from Definition 1) so that

$$\int_{t_0}^{t_n}\left[s^{(m+2p)}(t)\right]^2 dt=\int_a^b\left[s^{(m+2p)}(t)\right]^2 dt=$$

$$=(-1)^{m-1}\int_a^b s^{(2m+2p-1)}(t)\cdot s^{(2p-1)}(t)\,dt=$$

$$=(-1)^{m-1}\sum_{k=1}^n C_k\int_{t_{k-1}}^{t_k} s^{(2p+1)}(t)\,dt=$$

$$=(-1)^{m-1}\sum_{k=1}^n C_k\left(s^{(2p)}(t_k)-s^{(2p)}(t_{k-1})\right)=0,$$

where $C_k=s^{(2m+2p-1)}(t)\big|_{I_k}$ $k=\overline{1,n}$ (by Condition $2^0$ from Definition 1).

Therefore, $s^{(m+2p)}(t)=0$, for all $t\in[a,b]$.

Since $s\in\mathscr{P}_{m+2p-1}$ on $I_0\cup I_{n+1}$ it follows $s^{(m+2p)}(t)=0$ for any $t\in I_0\cup I_{n+1}$. By continuity of $s^{(m+2p)}$ on $\mathbb{R}$ it follows $s^{(m+2p)}(t)=0$ for all $t\in R$ (see the Condition $1^0$ from Definition 1). Then $s\in\mathscr{P}_{m+2p-1}$ on $\mathbb{R}$, implies $s^{(2p)}\in\mathscr{P}_{m-1}$ on $\mathbb{R}$. But $s^{(2p)}(t_k)=0$, $k=\overline{0,n}$ $(n>m)$ implies $s^{(2p)}(t)=0$ for all $t\in\mathbb{R}$ and, consequently, $s\in\mathscr{P}_{2p-1}$ on $\mathbb{R}$.

As $s^{(q)}(a)=s^{(q)}(b)=0$, $q=0,1,\dots,p-1$ we infer that $s\equiv 0$ or $\mathbb{R}$. But then all the coefficients of $s$ are null, so that the homogeneous system associated to (11) has only the null solution.

*Remark* 1. *By Theorem 2, if y is the exact solution of the differential equ a-tions (1) with condition (2), then there is only one function $s_y\in S_{2m+2p-1}(\Delta_n)$ verifying the conditions (2).*

Let

(12) $$H_2^{m+2p}([a,b]):=\{g:[a,b]\to\mathbb{R},\ g^{(m+2p-1)}\text{ absolutely continuous on }[a,b]\text{ and }g^{(m+2p)}\in L_2[a,b]\}.$$

and let $Y=\left(\alpha^{(0)},\alpha^{(1)},\dots,\alpha^{(p-1)},\beta^{(0)},\beta^{(1)},\dots,\beta^{(p-1)},\gamma_0,\gamma_1,\dots,\gamma_n\right)\in\mathbb{R}^{n+2p+1}$ be a fixed vector.

Denote

(13) $$H_2^{m+2p}(\Delta_n,Y):=\{g\in H_2^{m+2p}([a,b]): g^{(2p)}(t_k)=\gamma_k,\ k=0,1,2,\dots,n;\ g^{(q)}(a)=\alpha^{(q)},\ g^{(q)}(b)=\beta^{(q)}(b)=\beta^{(q)},\ q=\overline{0,p-1}\}$$



By Theorem 2, there is only one spline function $s_y\in S_{2m+2p-1}(\Delta_n)$ such that $s_y\in H_2^{m+2p}(\Delta_n,Y)$.

Furthermore, we have:

THEOREM 3. *([9], Th. 5 and Th. 6).*

*a) If $g\in H_2^{m+2p}(\Delta_n,Y)$ then*

(14) $$\left\|s_y^{(m+2p)}\right\|_2\le\left\|g^{(m+2p)}\right\|_2;$$

*b) If $f\in H_2^{m+2p}(\Delta_n)$ then*

(15) $$\left\|f^{(m+2p)}-s_f^{(m+2p)}\right\|_2\le\left\|f^{(m+2p)}-s^{(m+2p)}\right\|_2,$$

*for any $s\in S_{2m+2p-1}(\Delta_n)$ (Here $s_f$ is given by Theorem 2).*

*Proof.* To prove (14) we shall use the identity

$$\left\|g^{(m+2p)}-s_y^{(m+2p)}\right\|_2^2=\int_a^b\left[g^{(m+2p)}(t)-s_y^{(m+2p)}(t)\right]^2dt=$$

$$=\left\|g^{(m+2p)}\right\|_2^2-\left\|s_y^{(m+2p)}\right\|_2^2-2\int_a^b s_y^{(m+2p)}(t)\left[g^{(m+2p)}(t)-s_y^{(m+2p)}(t)\right]dt$$

where the last term is null. Indeed, integrating by parts, we find

$$\int_a^b s_y^{(m+2p)}(t)\left[g^{(m+2p)}(t)-s_y^{(m+2p)}(t)\right]dt=$$

$$=(-1)^{m-1}\sum_{k=1}^n C_k\left[\left(g^{(2p)}-s_y^{(2p)}\right)(t_k)-\left(g^{(2p)}-s_y^{(2p)}\right)(t_{k-1})\right]=0$$

where $C_k=s^{(2m+2p-1)}\big|_{I_k}$ $k=1,2,\dots,n$ and $s_y^{(m+2p+j)}(a)=s_y^{(m+2p+j)}(b)=0$ for $j=0,1,\dots,m-2$ (by Condition $3^0$ from Definition 1).

It follows

$$0\le\left\|g^{(m+2p)}\right\|_2^2-\left\|s_y^{(m+2p)}\right\|_2^2$$

implying the relation (14).

To prove (15) we shall use the identity

$$\left\|s^{(m+2p)}-f^{(m+2p)}\right\|_2^2=\left\|s^{(m+2p)}-s_f^{(m+2p)}\right\|_2^2+\left\|s_f^{(m+2p)}-f^{(m+2p)}\right\|_2^2+$$

$$+2\int_a^b\left[s^{(m+2p)}(t)-s_f^{(m+2p)}(t)\right]\left[s_f^{(m+2p)}(t)-f^{(m+2p)}(t)\right]dt$$

where, by integrating by parts, the last term is again null. To show this one uses the equalities

$$\left(s^{(m+2p+j)} - s_f^{(m+2p+j)}\right)(a) = \left(s^{(m+2p+j)} - s_f^{(m+2p+j)}\right)(b) = 0$$

for $j = 0, 1, 2, \ldots, m-2$, and

$$\left(s^{(m+2p-1)}(t) - s_f^{(m+2p-1)}(t)\right)\Big|_{I_k} = c_k(s), \quad k = \overline{1,n}$$

(constants depending on $s$). In conclusion

$$\left\|s^{(m+2p)} - f^{(m+2p)}\right\|_2^2 = \left\|s^{(m+2p)} - s_f^{(m+2p)}\right\|_2^2 + \left\|s_f^{(m+2p)} - f^{(m+2p)}\right\|_2^2 \tag{16}$$

which imply (15).

*Remark 2. By (15), we obtain for* $s \equiv 0$

$$\left\|f^{(m+2p)} - s_f^{(m+2p)}\right\|_2 \le \left\|f^{(m+2p)}\right\|_2. \tag{17}$$

Returning to the problem (1)–(2) we deduce

COROLLARY 4. *If the exact solution* $y$ *of the problem (1)–(2) is in* $H^{(m+2p)}([a,b])$ *and* $s_y \in S_{2m+p-1}(\Delta_n)$ *is the spline function associated to y, verifying the same boundary conditions as y, then the following evaluation*

$$\left\|y^{(m+2p)} - s_y^{(m+2p)}\right\|_2 \le \left\|y^{(m+2p)}\right\|_2 \tag{18}$$

*holds.*

THEOREM 5. *If* $y$ *is the exact solution of the problem (1)–(2),* $y \in H^{(m+2p)}([a,b])$ *and* $s_y \in S_{2m+p-1}(\Delta_n)$ *is the approximant spline function, then the following inequalities:*

$$\begin{aligned}&\left\|y^{(m+2p-l)} - s_y^{(m+2p-l)}\right\|_\infty \le \\ &\le \sqrt{m}(m-1)\cdots(m-l+1)\|\Delta_n\|^{l-\frac{1}{2}} \cdot \left\|y^{(m+2p)}\right\|_2\end{aligned} \tag{19}$$

*holds, for* $l = \{2, 3, \ldots, m\}$ and $\|\Delta_n\| = \max\{t_i - t_{i-1}, i = \overline{1,n}\}$.

*Proof.* We have

$$y^{(2p)}(t_i) - s_y^{(2p)}(t_i) = 0, \quad i = 0, 1, 2, \ldots, n.$$

By Rôlle's Theorem it follows the existence of the points $t_i^{(1)} \in (t_i, t_{i+1})$, $i = 0, 1, 2, \ldots, n-1$ such that

$$y^{(2p+1)}\left(t_i^{(1)}\right) - s_y^{(2p+1)}\left(t_i^{(1)}\right) = 0, \quad i = 0, 1, 2, \ldots, n-1.$$

Furthermore, we have



$$\left|t_i^{(1)} - t_{i+1}^{(1)}\right| \le 2\|\Delta_n\|, \ i = 0, 1, 2, \ldots, n-2.$$

Applying again Rôlle's Theorem for $y^{(2p+1)}$ one obtains the existence of the points $t_i^{(2)} \in \left(t_i^{(1)}, t_{i+1}^{(1)}\right)$, $i = 0, 1, 2, \ldots, n-2$ such that

$$y^{(2p+2)}\left(t_i^{(2)}\right) - s_y^{(2p+2)}\left(t_i^{(2)}\right) = 0, \ i = 0, 1, 2, \ldots, n-2$$

and

$$\left|t_i^{(2)} - t_{i+1}^{(2)}\right| \le 3\|\Delta_n\|, \ i = 0, 1, 2, \ldots, n-3.$$

A $k$-times applications of Rôlle's Theorem yields the points

$$t_i^{(k)} \in \left(t_i^{(k-1)}, t_{i+1}^{(k-1)}\right), \quad i = \overline{0, n-k}, \ k = 1, 2, \ldots, m-1$$

such that

$$y^{(2p+k)}\left(t_i^{(k)}\right) - s_y^{(2p+k)}\left(t_i^{(k)}\right) = 0$$

and

$$\left|t_i^{(k)} - t_{i+1}^{(k)}\right| \le (k+1)\|\Delta_n\|, \quad i = \overline{0, n-k} \text{ and } k = 1, 2, \ldots, n-1.$$

For $k = m-1$ we obtain

$$y^{(m+2p-1)}\left(t_i^{(m-1)}\right) - s_y^{(m+2p-1)}\left(t_i^{(m-1)}\right) = 0, \quad i = \overline{0, n-m+1}$$

and

$$\left|t_i^{(m-1)} - t_{i+1}^{(m-1)}\right| \le m\|\Delta_n\|, \quad i = \overline{0, n-m+1}.$$

Since $\left|a - t_0^{(m-1)}\right| < m\|\Delta_n\|$ and $\left|b - t_{n-m+1}^{(m-1)}\right| < \|\Delta_n\|$ it follows that for every $t \in [a, b]$ there is $i_0 \in \{0, 1, \ldots, n-m+1\}$ such that

$$\left|t - t_{i_0}^{(m-1)}\right| \le m\|\Delta_n\|$$

and

$$\begin{aligned}\left|y^{(m+2p-1)}(t) - s_y^{(m+2p-1)}(t)\right| &= \left|\int_{t_{i_0}^{(m-1)}}^{t}\left[y^{(m+2p)}(u) - s_y^{(m+2p)}(u)\right]\right| \le \\ &\le \left(\int_{t_{i_0}^{(m-1)}}^{t} \mathrm{d}u\right)^{\frac{1}{2}} \cdot \left(\int_{t_{i_0}^{(m-1)}}^{t}\left(y^{(m+2p)}(u) - s_y^{(m+2p)}(u)\right)^2 \mathrm{d}u\right)^{\frac{1}{2}} \le \\ &\le \sqrt{m\|\Delta_n\|} \cdot \left\|y^{(m+2p)} - s_y^{(m+2p)}\right\|_2 \le \sqrt{m}\,\|\Delta_n\|^{\frac{1}{2}}\left\|y^{(m+2p)}\right\|_2\end{aligned}$$

(according to (17)).

Here from we deduce

$$\left\|y^{(m+2p-1)}-s_y^{(m+2p-1)}\right\|_\infty \le \sqrt{m}\cdot\|\Delta_n\|^{\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2 .$$

Similarly, for every $t\in[a,b]$ there exists $i_0\in\{0,1,\dots,n-m+2\}$ such that

$$\left|t-t_{i_0}^{(m-2)}\right|\le (m-1)\|\Delta_n\|$$

so that

$$\left|y^{(m+2p-2)}(t)-s_y^{(m+2p-2)}(t)\right| = \left|\int_{t_{i_0}^{(m-2)}}^{t} \left[y^{(m+2p-1)}(u)-s_y^{(m+2p-1)}(u)\right]du\right| \le$$

$$\le \left|t-t_{i_0}^{(m-2)}\right|\cdot \left\|y^{(m+2p-1)}-s_y^{(m+2p-1)}\right\|_\infty \le$$

$$\le \sqrt{m}(m-1)\|\Delta_n\|^{1+\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2 .$$

It follows

$$\left\|y^{(m+2p-2)}-s_y^{(m+2p-2)}\right\|_\infty \le \sqrt{m}(m-1)\|\Delta_n\|^{1+\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2 .$$

In general we find

$$\left\|y^{(m+2p-l)}-s_y^{(m+2p-l)}\right\|_\infty\le\sqrt{m}(m-1)\cdots(m-l+1)\|\Delta_n\|^{l-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2$$

for all $l=2,3,4,\dots,m$.

*Remark* 3. *For $l = m$ we find*

(20) $$\left\|y^{(2p)}-s_y^{(2p)}\right\|_\infty\le\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2 .$$

In the following we shall give estimations for the norms

(21) $$\left\|y^{(q)}-s_y^{(q)}\right\|_\infty,\quad q=0,1,2,\dots,2p-1,$$

necessary for the numerical treatment of the problem (1)–(2).

COROLLARY 6. *If the exact solution $y$ of the problem (1)–(2) belongs to $H_2^{m+2p}([a,b])$ and $s_y\in S_{2m+2p-1}(\Delta_n)$ is the associated spline solution, then the following estimation hold:*

$$\left\|y^{(q)}-s_y^{(q)}\right\|_\infty\le(b-a)^{2p-q}\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2$$

*for $q = 0, 1, 2,\dots, 2p-1$.*

*Proof.* Since $y(t_0)-s_y(t_0)=y(t_n)-s_y(t_n)=0$, it follows that there exists at least one point $t_0^{(1)}\in(t_0,t_n)$ such that



$$y'\left(t_0^{(1)}\right)-s_y'\left(t_0^{(1)}\right)=0,$$

implying

$$y'(t_0)-s_y'(t_0)=y'\left(t_0^{(1)}\right)-s_y'\left(t_0^{(1)}\right)=y'(t_n)-s_y'(t_n)=0.$$

Then it will exist the points $t_0^{(2)}\in\left(t_0,t_0^{(1)}\right)$, $t_1^{(2)}\in\left(t_0^{(1)},t_n\right)$, $t_0<t_0^{(2)}<t_1^{(2)}<t_n$ such that

$$y''(t_0)-s_y''(t_0)=y''\left(t_0^{(2)}\right)-s_y''\left(t_0^{(2)}\right)=y''\left(t_1^{(2)}\right)-s_y''\left(t_1^{(2)}\right)=$$

$$y''(t_n)-s_y''(t_n)=0.$$

In general, for every $q\in\{2,\dots,p-1\}$ there are the points $t_0^{(q)}\in\left(t_0,t_0^{(q-1)}\right)$, $t_1^{(q)}\in\left(t_0^{(q-1)},t_1^{(q-1)}\right),\dots,t_{q-1}^{(q)}\in\left(t_{q-2}^{(q-1)},t_n\right)$, $t_0<t_0^{(q)}<t_1^{(q)}<\dots<t_{q-1}^{(q)}<t_n$ such that

$$y^{(q)}(t_0)-s_y^{(q)}(t_0)=y^{(q)}\left(t_0^{(q)}\right)-s_y^{(q)}\left(t_0^{(q)}\right)=\dots$$

$$\dots=y^{(q)}\left(t_{q-1}^{(q)}\right)-s_y^{(q)}\left(t_{q-1}^{(q)}\right)=y^{(q)}(t_n)-s_y^{(q)}(t_n)=0.$$

For $q = p-1$ we deduce the existence of $p+1$ distinct points

$$t_0<t_0^{(p-1)}<t_1^{(p-1)}<\dots<t_{p-2}^{(p-1)}<t_n$$

at which the $(p-1)$ - derivative of the difference $y(t)-s_y(t)$ vanishes.

Finally, we deduce the existence of a point $\bar{t}_1\in(a,b)$ such that

$$y^{(2p-1)}(\bar{t}_1)-s_y^{(2p-1)}(\bar{t}_1)=0.$$

But then, for all $t\in[a,b]$, we have

$$\left|y^{(2p-1)}(t)-s_y^{(2p-1)}(t)\right|=\left|\int_{\bar{t}_1}^{t}\left[y^{(2p)}(h)-s_y^{(2p)}(h)\right]dh\right|\le$$

$$\le\left|t-\bar{t}_1\right|\cdot\left\|y^{(2p)}-s_y^{(2p)}\right\|_\infty$$

so that

$$\left\|y^{(2p-1)}-s_y^{(2p-1)}\right\|_\infty\le(a-b)\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2 .$$

Similarly, there is $\bar{t}_2\in(a,b)$ such that

$$\left|y^{(2p-2)}(t)-s_y^{(2p-2)}(t)\right|=\left|\int_{\bar{t}_2}^{t}\left[y^{(2p-1)}(h)-s_y^{(2p-1)}(h)\right]dh\right|\le$$

$$\le(b-a)\left\|y^{(2p-1)}-s_y^{(2p-1)}\right\|_\infty$$

where from

$$\left\|y^{(2p-2)}-s_y^{(2p-2)}\right\|_\infty \le (b-a)^2\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2.$$

Continuing in this manner, we obtain

$$\left\|y^{(2p-l)}-s_y^{(2p-l)}\right\|_\infty \le (b-a)^l\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2$$

for $l=0,1,2,\dots,2p-1$.

Therefore

$$\left\|y^{(q)}-s_y^{(q)}\right\|_\infty \le (b-a)^{2p-q}\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\cdot\left\|y^{(m+2p)}\right\|_2,$$

for $q=0,1,2,\dots,2p-1$

which ends the proof.

*Remark* 4. *For $q=0$ one obtains*

$$\left\|y-s_y\right\|_\infty \le (b-a)^{2p}\sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}}\left\|y^{(m+2p)}\right\|_2.$$

*By Definition 1, $m\ge 2$ and then $\left\|y-s_y\right\|_\infty$ is at least of order*

$$O\left(\|\Delta_n\|^{\frac{3}{2}}\right).$$

*Example.*

Consider the problem

$$(P1)\quad \begin{cases} y^{(4)}(t)=(t^4+14t^3+49t^2+32t-12)\,e^t, & t\in[0,1]\\ y(0)=y'(0)=y(1)=y'(1)=0 \end{cases}$$

Problem $(P1)$ is a problem of Karpilovskaja type for a fourth order differential equation which is studied also in [12].

In Table 1 the maximum values of the error at the nodes of the uniform partition $\Delta_n$ : $n=5, 10, 20, 30, 40$ are presented

*Table 1*

| n | maximum values of the error at the nodes of $\Delta_n$ |
|---|---|
| 5 | 0.0000380786 |
| 10 | 0.0000023077 |
| 20 | 0.0000001162 |
| 30 | 0.0000000198 |
| 40 | 0.0000000055 |



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