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# ON A POSSSIBLE DETERMINATION OF THE FRAME BOUNDS

DANIELA CĂTINAŞ

**Abstract.** We give conditions which ensure that the subset $\{x_n, n \in \mathbb{N}^*\}$ of a separable real Hilbert space $\mathcal{H}$ is a frame, and we obtain formulas for the frame bounds in terms of the eigenvalues of the Gram matrices of the finite subsets.

## 1. INTRODUCTION

In a separable Hilbert space $\mathcal{H}$, a subset $\{e_n, n \in \mathbb{N}\}$ is called a *frame* if there exists $A, B, B > 0, A < \infty$ (called the frame bounds) such that $B\,\|x\|^2 \leqslant \sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \leqslant A\|x\|^2$, for every $x \in \mathcal{H}$. For such a sequence, we can find the set $\{\tilde{e}_n, n \in \mathbb{N}\}$ (called the *dual frame*) having the bounds $A^{-1}, B^{-1}$, and allowing the reconstruction $x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle \tilde{e}_n = \sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle e_n$, for every $x \in \mathcal{H}$ (see [2]). The advantage of the frames over the orthonormal and complete bases (which allow the Fourier expansion $x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$, $\forall x \in \mathcal{H}$) is that the set $\{e_n, n \in \mathbb{N}\}$ need be neither orthonormal nor linearly independent. Moreover, if $A = B$ (tight frame), than that frame allows the unique expansion $x = A^{-1} \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$, $\forall x \in \mathcal{H}$, similarly to the Fourier one.

## 2. PRELIMINARIES

In this section we remind some known relations which we shall use in the following.

Let $\{x_n, n \in \mathbb{N}^*\}$ be a subset of the separable real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot\rangle)$, $x \in \mathcal{H}$ and $\langle \cdot, \cdot\rangle_e$ the standard Euclidean product in $\mathbb{R}^n$. The Gram matrices

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associated to $\{x_n, n \in \mathbb{N}^*\}$, defined by

$$G_n = G(x_1,\dots,x_n) = \begin{pmatrix} \langle x_1,x_1\rangle & \langle x_1,x_2\rangle & \dots & \langle x_1,x_n\rangle \\ \langle x_2,x_1\rangle & \langle x_2,x_2\rangle & \dots & \langle x_2,x_n\rangle \\ \vdots & & & \\ \langle x_n,x_1\rangle & \langle x_n,x_2\rangle & \dots & \langle x_n,x_n\rangle \end{pmatrix}, \ n \in \mathbb{N}^*,$$

have the following properties:

**P1:** All the eigenvalues of the matrices $G_n$ are nonnegative numbers; if the set $\{x_1,\dots,x_n\}$ is linearly independent, then these eigenvalues are positive.

**P2:** The system

$$\text{(2.1)} \qquad \begin{cases} c_1^n\langle x_1,x_1\rangle + c_2^n\langle x_1,x_2\rangle + \dots + c_n^n\langle x_1,x_n\rangle = \langle x_1,x\rangle \\ \vdots \\ c_1^n\langle x_n,x_1\rangle + c_2^n\langle x_n,x_2\rangle + \dots + c_n^n\langle x_n,x_n\rangle = \langle x_n,x\rangle \end{cases}$$

with the unknowns $c_1^n, c_2^n, \dots, c_n^n$ is solvable for every $x \in \mathcal{H}$, since if a row of the matrix of coefficients is a linear combination of the other rows, then the same thing happens in the augmented matrix.

If rank $G_n = p = p(n)$, (for example $x_{\tau_n(1)},\dots,x_{\tau_n(p)}$ are linearly independent, where $\tau_n$ is a permutation of the set $\{1,2,\dots,n\}$, and $x_{\tau_n(p+1)},\dots,x_{\tau_n(n)}$ are linear combinations of them), then we will consider the solution $(c_1^n, c_2^n,\dots,c_n^n)$ with $(c_{\tau_n(1)}^n,\dots,c_{\tau_n(p)}^n)$ as the solution of the linear system

$$\text{(2.2)} \qquad G_{\tau_n(p)} \begin{pmatrix} c_1^n \\ \vdots \\ c_p^n \end{pmatrix} = \begin{pmatrix} \langle x_{\tau_n(1)},x\rangle \\ \vdots \\ \langle x_{\tau_n(p)},x\rangle \end{pmatrix}$$

and $c_{\tau_n(p+1)}^n = \dots = c_{\tau_n(n)}^n = 0$. (We have denoted $G_\tau(p) = G(x_{\tau(1)},\dots,x_{\tau(p)})$.

Denoting by $\lambda_n^{\min}$ and $\lambda_n^{\max}$ the smallest, respective the largest eigenvalue of the Gram matrix $G_n$, then the following inequalities hold:

**P3:** $\lambda_n^{\min}\langle y,y\rangle_e \le \langle G_n y,y\rangle_e \le \lambda_n^{\max}\langle y,y\rangle_e$, $\forall y \in \mathbb{R}^n$

**P4:** $\lambda_n^{\min}\langle G_n y,y\rangle_e \le \langle G_n y,G_n y\rangle_e \le \lambda_n^{\max}\langle G_n y,y\rangle_e$, $\forall y \in \mathbb{R}^n$

*Proof.* $\langle G_n y,G_n y\rangle_e - \lambda_n^{\max}\langle G_n y,y\rangle_e = \langle G_n^2 y,y\rangle_e - \langle\lambda_n^{\max}G_n y,y\rangle_e = \langle(G_n^2 - \lambda_n^{\max}G_n)y,y\rangle_e$.

It can be easily proved that if $A$ is a symmetric matrix having the diagonal form $B$, diag$B = (\lambda_n^{\min},\dots,\lambda_n^{\max})$ and $P$ is a polynomial, then the matrix $P(A)$ has the diagonal form $C$, with diag$C = (P(\lambda_n^{\min}),\dots,P(\lambda_n^{\max}))$.

Hence the diagonal form of the matrix $G_n^2 - \lambda_n^{\max}G_n$ is

$$\mathrm{diag}(G_n^2 - \lambda_n^{\max}G_n) = (\lambda_n^{\min}(\lambda_n^{\min} - \lambda_n^{\max}),\dots,\lambda_n^{\max}(\lambda_n^{\max} - \lambda_n^{\max})).$$



All its eigenvalues are nonpositive numbers, so $\langle(G_n^2 - \lambda_n^{\max}G_n)y,y\rangle_e$ is a negative definite quadratic form, whence the stated inequality.

In the same way, $\langle G_n y,G_n y\rangle_e - \lambda_n^{\min}\langle G_n y,y\rangle_e = \langle(G_n^2 - \lambda_n^{\min}G_n)y,y\rangle_e$. The diagonal form of the matrix $G_n^2 - \lambda_n^{\min}G_n$ is

$$\mathrm{diag}(G_n^2 - \lambda_n^{\min}G_n) = (\lambda_n^{\min}(\lambda_n^{\min} - \lambda_n^{\min}),\dots,\lambda_n^{\max}(\lambda_n^{\max} - \lambda_n^{\min})).$$

All its eigenvalues are nonnegative numbers, so $\langle(G_n^2 - \lambda_n^{\min}G_n)y,y\rangle_e$ is a positive definite quadratic form, whence the stated inequality. □

We will study the set $\{x_n : n \in \mathbb{N}^*\}$, which may be linearly dependent.

### 3. PROPERTIES OF THE SUM $\langle x_1,x\rangle^2 + \dots + \langle x_n,x\rangle^2$, $x \in \mathcal{H}$, $x \neq 0$

Consider in the beginning a fixed $n \in \mathbb{N}^*$.

Let $C_n = (c_1^n, c_2^n,\dots,c_n^n)$ and $X_n = \begin{pmatrix} \langle x_1,x\rangle \\ \vdots \\ \langle x_n,x\rangle \end{pmatrix}$. The system (2.1) may be written as $G_n C_n^T = X_n$.

If we suppose that $x_{\tau_n(1)},\dots,x_{\tau_n(p)}$ are linearly independent and $x_{\tau_n(k)}$, $k = \overline{p+1,n}$, are linear combinations of them ($p = p(n)$), the solution of the system (2.1), considered in **P2**, may be written as

$$\text{(3.1)} \qquad \begin{cases} C_{p(n)}^T = G_{p(n)}^{-1}X_{p(n)} \\ c_{\tau_n(p+1)}^n = \dots = c_{\tau_n(n)}^n = 0 \end{cases}$$

where

$$\text{(3.2)} \qquad G_{p(n)} = G(x_{\tau_n(1)},\dots,x_{\tau_n(p)}) \text{ and } X_{p(n)} = \begin{pmatrix} \langle x_{\tau_n(1)},x\rangle \\ \vdots \\ \langle x_{\tau_n(p)},x\rangle \end{pmatrix}.$$

Consider the expression

$$E(X_n) = c_1^n\langle x_1,x\rangle + \dots + c_n^n\langle x_n,x\rangle = C_n X_n.$$

In the case when $(c_1^n,\dots,c_n^n)$ is the solution (3.1) of the system (2.1), it becomes the quadratic form

$$E(X_{p(n)}) = C_n X_n = \langle G_{p(n)}^{-1}X_{p(n)}, X_{p(n)}\rangle_e.$$

The eigenvalues of the matrix $G_{p(n)}^{-1}$ are the inverse of the eigenvalues of $G_{p(n)}$, (which are positive numbers), so, taking into account **P3**, we shall have that:

$$\text{(3.3)} \qquad E(X_{p(n)}) = \langle G_{p(n)}^{-1}X_{p(n)}, X_{p(n)}\rangle_e \le \frac{1}{\lambda_n^{\min}(p)}\langle X_{p(n)}, X_{p(n)}\rangle_e,$$

$\lambda_n^{\min}(p)$ being the smallest eigenvalue of the matrix $G_{p(n)}$.

*Remark* 1. Instead of the linearly independent elements $x_{\tau_n(1)},\dots,x_{\tau_n(p)}$, we may take other linearly independent elements $x_{\sigma_n(1)},\dots,x_{\sigma_n(p)}$ with $\sigma_n$ permutation of $\{1,2,\dots,n\}$, different from $\tau_n$. Hence, the matrix $G_{p(n)}$ is not unique, so it is possible to find more values for $\lambda_n^{\min}(p)$. We will choose the largest of them.

CONSEQUENCES:

**1.** Let $(c_1^n,\dots,c_n^n)$ be an arbitrary solution of the system (2.1). According to **P4** we obtain:

$$
\begin{aligned}
\sum_{k=1}^{n}\langle x_k,x\rangle^2 &= \sum_{k=1}^{n}(c_1^n\langle x_1,x\rangle+\dots+c_n^n\langle x_n,x\rangle^2)\\
&= \langle G_nC_n^T,G_nC_n^T\rangle_e\le\lambda_n^{\max}\langle G_nC_n^T,C_n^T\rangle_e\\
&= \lambda_n^{\max}\sum_{k=1}^{n}c_k^n(c_1^n\langle x_k,x_1\rangle+c_2^n\langle x_k,x_2\rangle+\dots+c_n^n\langle x_k,x_n\rangle) \qquad (3.4)\\
&= \lambda_n^{\max}\sum_{k=1}^{n}c_k^n\langle x_k,x\rangle.
\end{aligned}
$$

*Remark* 2. When $p(n)=n$, the above inequality can be immediately obtained: from **P3** it follows that

$$\frac{1}{\lambda_n^{\max}}\langle X_n,X_n\rangle_e\le\langle G_n^{-1}X_n,X_n\rangle_e=E(X_n),$$

$$\text{i.e. }\sum_{k=1}^{n}\langle x_k,x\rangle^2\le\lambda_n^{\max}\sum_{k=1}^{n}c_k^n\langle x_k,x\rangle.$$

**2.** From (3.3) we obtain:

$$\langle X_n,X_n\rangle\ge\langle X_{p(n)},X_{p(n)}\rangle\ge\lambda_n^{\min}(p)E(X_{p(n)}),\ \text{i.e.}$$

$$\sum_{k=1}^{n}\langle x_k,x\rangle^2\ge\sum_{k=1}^{p}\langle x_{\tau_n(k)},x\rangle^2\ge\lambda_n^{\min}(p)\sum_{k=1}^{p}c_{\tau_n(k)}^n\langle x_{\tau_n(k)},x\rangle, \qquad (3.5)$$

where $(c_1^n,\dots,c_n^n)$ is the solution of the system (2.1) defined by (3.1).

## 4. PROPERTIES OF THE SUM $\sum_{k=1}^{n}c_k\langle x_k,x\rangle$, $x\in\mathcal{H}$, $x\neq 0$, $(c_1,\dots,c_n)$ SOLUTION OF THE SYSTEM (2.1)

Suppose a fixed $n\in\mathbb{N}^*$ and $p$ defined at the beginning of section 2.



THEOREM 1. *Let* $F_n:\mathbb{R}^n\to\mathbb{R}$, $F_n(t_1,\dots,t_n)=\langle x-\sum_{k=1}^{n}t_kx_k,x-\sum_{k=1}^{n}t_kx_k\rangle$, *and* $(c_1^n,\dots,c_n^n)$ *an arbitrary solution of the system (2.1). Then:*

$1^o$. *If* $p=n$, *then* $\min F_n=\langle x,x\rangle-\sum_{k=1}^{n}c_n^k\langle x_k,x\rangle$.

$2^o$. *If* $p<n$, *then* $\min F_n=\langle x,x\rangle-\sum_{k=1}^{p}c_{\tau_n(k)}^n\langle x_{\tau_n(k)},x\rangle=\langle x,x\rangle-\sum_{k=1}^{n}c_k^n\langle x_k,x\rangle$.

*Proof.* $1^o$. The necessary conditions for extremum, $\frac{\partial F_n}{\partial t_k}=0$, $k=\overline{1,n}$, lead to the system (2.1). The Hessian matrix of the function $F_n$ is, at any point, the Gram matrix $G(x_1,\dots,x_n)$, which is strictly positive defined cf. **P1**. So, $F_n$ will have a minimum attained at $(c_1^n,\dots,c_n^n)$, the solution of the system (2.1), namely:

$$\min F_n=\langle x,x\rangle-\sum_{k=1}^{n}c_k^n\langle x_k,x\rangle.$$

□

$2^o$. First we prove the following auxiliary results.

LEMMA 1. *Let* $(c_{\tau_n(1)}^n,\dots,c_{\tau_n(p)}^n)$ *be the solution of the system (2.2) and* $(d_1^1,\dots,d_n^n)$ *an arbitrary solution of the system (2.1).*

*Then* $\sum_{k=1}^{n}d_k^n\langle x_k,x\rangle=\sum_{k=1}^{p}c_{\tau_n(k)}^n\langle x_{\tau_n(k)},x\rangle$, *i.e. then value of the function* $F_n$ *is the same at every stationary point:* $\langle x,x\rangle-\sum_{k=1}^{p}c_{\tau_n(k)}^n\langle x_{\tau_n(k)},x\rangle$.

*Proof.* For the sake of simplicity we omit the upper indices and we consider that $x_1,\dots,x_p$ are linearly independent and $x_{p+1},\dots,x_n$ are linear combinations of them:

$$x_{p+j}=\sum_{k=1}^{p}\alpha_{jk}x_k,\ j=\overline{1,n-p},\ \text{with }\alpha_{jk}\in\mathbb{R}.$$

Then the solution of the system (2.2) can be written as $(c_1,\dots,c_p)$.

The system (2.1) becomes:

$$\begin{cases} \langle x_1,x_1\rangle d_1+\ldots+\langle x_1,x_p\rangle d_p=\langle x_1,x_1\rangle-\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_1,x_k\rangle\\ \vdots\\ \langle x_p,x_1\rangle d_1+\ldots+\langle x_p,x_p\rangle d_p=\langle x_1,x_p\rangle-\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_p,x_k\rangle\\ \vdots\\ \langle x_n,x_1\rangle d_1+\ldots+\langle x_n,x_p\rangle d_p=\langle x_1,x_n\rangle-\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_n,x_k\rangle \end{cases}$$

Its general solution will be $(d_1,\ldots,d_n)$, with $d_i=\frac{\det A_i}{\det G_p}$, $i=\overline{1,p}$ and $d_{p+1}$, $\ldots,d_n$ arbitrary, the matrices $A_i$, $i=\overline{1,p}$, being given by

$$A_i=\begin{pmatrix} \langle x_1,x_1\rangle\ldots\langle x_1,x_{i-1}\rangle & \langle x,x_1\rangle-\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_1,x_k\rangle & \langle x_1,x_{i+1}\rangle\ldots\langle x_1,x_p\rangle\\ \vdots & & \vdots\\ \langle x_p,x_1\rangle\ldots\langle x_p,x_{i-1}\rangle & \langle x,x_p\rangle-\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_p,x_k\rangle & \langle x_p,x_{i+1}\rangle\ldots\langle x_p,x_p\rangle \end{pmatrix}$$

We get

$$\begin{aligned}\sum_{k=1}^{n}d_k\langle x_k,x\rangle &= \sum_{i=1}^{p}d_i\langle x_i,x\rangle+\sum_{i=1}^{n-p}d_{p+i}\langle x_{p+i},x\rangle=\\ &= \frac{1}{\det G_p}\sum_{i=1}^{p}(\det A_i)\langle x_i,x\rangle+\sum_{j=1}^{n-p}d_{p+j}\sum_{k=1}^{p}\alpha_{jk}\langle x_k,x\rangle.\end{aligned}$$

If we split $\det A_i$ after the column $i$ we obtain:

$$\begin{aligned}\det A_i &= \begin{vmatrix} \langle x_1,x_1\rangle\ldots\langle x_1,x_{i-1}\rangle & \langle x,x_1\rangle & \langle x_1,x_{i+1}\rangle\ldots\langle x_1,x_p\rangle\\ \vdots & & \\ \langle x_p,x_1\rangle\ldots\langle x_p,x_{i-1}\rangle & \langle x,x_p\rangle & \langle x_p,x_{i+1}\rangle\ldots\langle x_p,x_p\rangle \end{vmatrix}\\ &- \sum_{j=1}^{n-p}d_{p+j}\alpha_{ji}\det G_p=\left(c_i-\sum_{j=1}^{n-p}d_{p+j}\alpha_{ji}\right)\det G_p.\end{aligned}$$



So, $\sum\limits_{i=1}^{n}d_i\langle x_i,x\rangle=\sum\limits_{i=1}^{p}\left(c_i-\sum\limits_{j=1}^{n-p}d_{p+j}\alpha_{ji}\right)\langle x_i,x\rangle+\sum\limits_{j=1}^{n-p}d_{p+j}\sum\limits_{k=1}^{p}\alpha_{jk}\langle x_k,x\rangle=$ $\sum\limits_{i=1}^{p}c_i\langle x_i,x\rangle$.

Finally, the value of $F_n$ at any stationary point is

$\langle x,x\rangle-\sum\limits_{k=1}^{p}c^n_{\tau_n(k)}\langle x_{\tau_n(k)},x\rangle$. □

LEMMA 2. *The stationary points of $F_n$ are points of minimum.*

*Proof.* The function $F_n$ is in fact a second order polynomial with $n$ variables. Writing the Taylor formula at an arbitrary stationary point $(d_1,\ldots,d_n)$ we get (taking into account the fact that the Hessian matrix of $F_n$ is the Gram matrix $G_n$):

$$\begin{aligned}F_n(t_1,\ldots,t_n) &= F_n(d_1,\ldots,d_n)+\sum_{k=1}^{n}\frac{\partial F_n}{\partial t_k}(d_1,\ldots,d_n)(t_k-d_k)+\\ &+ Y^TG_n(x_1,\ldots,x_n)Y,\ Y=(t_1-d_1,\ldots,t_n-d_n)^T.\end{aligned}$$

Since $Y^TG_nY\geq 0$ (cf. **P1**, the matrix $G_n$ is positive defined), we obtain:

$$F_n(t_1,\ldots,t_n)\geq F_n(d_1,\ldots,d_n),\quad \forall(t_1,\ldots,t_n)\in\mathbb{R}^n,$$

i.e. $(d_1,\ldots,d_n)$ is a point of minimum. □

The two lemmas show that the minimum value of $F_n$ is

$$\langle x,x\rangle-\sum_{k=1}^{p}c^n_{\tau_n(k)}\langle x_{\tau_n(k)},x\rangle.$$

A consequence of lemma 1 is the following equality:

$$\sum_{k=1}^{p}c^n_{\tau_n(k)}\langle x_{\tau_n(k)},x\rangle=\sum_{k=1}^{n}c^n_k\langle x_k,x\rangle, \tag{4.1}$$

whence the stated affirmation 2°.

Relations (4.1) and (3.5) imply:

$$\sum_{k=1}^{n}\langle x_k,x\rangle^2\geq\lambda_n^{\min}(p)\sum_{k=1}^{n}c^n_k\langle x_k,x\rangle,\quad \forall n\in\mathbb{N}^*. \tag{4.2}$$

On the other hand, by **P4** we obtain, using the same equalities as in the consequence 1, section 2, that

$$\sum_{k=1}^{n}\langle x_k,x\rangle^2\geq\lambda_n^{*\min}\sum_{k=1}^{n}c^n_k\langle x_k,x\rangle,\quad \forall n\in\mathbb{N}^*. \tag{4.3}$$

where $\lambda_n^{*\min}$ is the smallest positive eigenvalue of the matrix $G_n$.

## 5. MAIN RESULT

THEOREM 2. *Let $\{x_n, n \in \mathbb{N}^*\}$ be a subset of the separable real Hilbert space $\mathcal{H}$ and $G_n$ the Gram matrices associated to sets $\{x_k, k = 1, \dots, n\}, n \in \mathbb{N}^*$. We denote by $\lambda_n^{\min} = \max\{\lambda_n^{\min}(p), \lambda_n^{*\min}\}$ and $\lambda_n^{\max}$ the largest eigenvalue of $G_n$.*

*Let $A = \limsup\limits_{n\to\infty} \lambda_n^{\max}$ and $B = \liminf\limits_{n\to\infty} \lambda_n^{\min}$.*

*The following statements are true:*

**1.** *If $A < \infty$, then $\sum\limits_{k=1}^{\infty} \langle x_k, x\rangle^2 \le A\|x\|^2\ \forall x \in \mathcal{H}$.*

**2.** *If $\overline{\operatorname{span}\{x_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$ and $B > 0$, then $B\|x\|^2 \le \sum\limits_{n=1}^{\infty} \langle x_n, x\rangle^2$, $\forall x \in \mathcal{H}$.*

**3.** *If $A < \infty$, $B > 0$ and $\overline{\operatorname{span}\{x_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$, then the set $\{x_n,\ n \in \mathbb{N}^*\}$ forms a frame in $\mathcal{H}$.*

*Proof.* **1.** Let $x \in \mathcal{H}$ and $\varepsilon > 0$. There exists $n_0(\varepsilon)$ such that $\lambda_n^{\max} < A + \varepsilon$, $\forall n \ge n_0(\varepsilon)$. From (3.4) it follows that $\sum\limits_{k=1}^{n} \langle x_k, x\rangle^2 \le (A+\varepsilon) \sum\limits_{k=1}^{n} c_k^n \langle x_k, x\rangle$, $\forall n \ge n_0(\varepsilon)$, where $(c_1^n, \dots, c_n^n)$ is an arbitrary solution of (2.1). Take an arbitrary $n \ge n_0(\varepsilon)$. Since $F_n(t_1, \dots, t_n) \ge 0$, $\forall (t_1, \dots, t_n) \in \mathbb{R}^n$, we have

$$\langle x, x\rangle \ge \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle \ge \frac{1}{A+\varepsilon} \sum_{k=1}^{n} \langle x_k, x\rangle^2.$$

Keeping the extreme sides and passing to limit we get

$$\langle x, x\rangle \ge \frac{1}{A+\varepsilon} \sum_{k=1}^{\infty} \langle x_k, x\rangle^2,$$

inequality which holds for every $\varepsilon > 0$.

So

$$\sum_{k=1}^{\infty} \langle x_k, x\rangle^2 \le A|x|^2. \tag{5.1}$$

**2.** Assume that the set $\{x_n, n \in \mathbb{N}^*\}$ is closed in $\mathcal{H}$, $(\overline{\operatorname{span}\{x_n, n \in \mathbb{N}^*\}} = \mathcal{H})$, and let $x \in \mathcal{H}$, $\varepsilon > 0$. Then there exists $n(\varepsilon) \in \mathbb{N}^*$ and $(c_k^*)_{k=\overline{1,n(\varepsilon)}}$ such that $F_n(c_1^*, \dots, c_n^*) < \varepsilon$, $\forall n \ge n(\varepsilon)$.



From (4.2) and (4.3) it follows that there exists $n_1(\varepsilon)$ such that

$$\sum_{k=1}^{n} \langle x_k, x\rangle^2 \ge (B-\varepsilon) \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle,\ \forall n \ge n_1(\varepsilon), \tag{5.2}$$

where $(c_1^n, \dots, c_n^n)$ is an arbitrary solution of (2.1). Consider now $n = \max\{n(\varepsilon), n_1(\varepsilon)\}$. Taking into account Theorem 1, we have $F_n(c_1^n, \dots, c_n^n) \le F_n(c_1^*, \dots, c_n^*) < \varepsilon$, i.e. $\sum\limits_{k=1}^{n} c_k^n \langle x_k, x\rangle > \langle x, x\rangle - \varepsilon$.

Consequently, by (5.2),

$$\sum_{n=1}^{\infty} \langle x_n, x\rangle^2 \ge \sum_{k=1}^{n} \langle x_k, x\rangle^2 \ge (B-\varepsilon) \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle \ge (B-\varepsilon)(\langle x, x\rangle - \varepsilon).$$

Keeping only the inequality $\sum\limits_{n=1}^{\infty} \langle x_n, x\rangle^2 \ge (B+\varepsilon)(\langle x, x\rangle - \varepsilon)$, which holds for every $\varepsilon > 0$, it follows

$$\sum_{n=1}^{\infty} \langle x_n, x\rangle^2 \ge B\|x\|^2. \tag{5.3}$$

**3.** The statement is an immediate consequence of the previous two affirmations. □

## 6. A PARTICULAR CASE

In the separable real Hilbert space $\mathcal{H}$ we consider the orthonormal set $\{p_n, n \in \mathbb{N}^*\}$. We construct the set $\{e_n, n \in \mathbb{N}^*\}$ in the following way:

$$
\begin{aligned}
e_1 &= a_{11}^1 p_1\\
e_2 &= a_{21}^1 p_1 + a_{22}^1 p_2\\
&\vdots\\
e_k &= a_{k1}^1 p_1 + \dots + a_{kk}^1 p_k\\
e_{k+1} &= a_{11}^2 p_{k+1}\\
e_{k+2} &= a_{21}^2 p_{k+1} + \dots + a_{22}^2 p_{k+2}\\
&\vdots\\
e_2k &= a_{k1}^2 p_{k+1} + \dots + a_{kk}^2 p_2 k\\
&\vdots
\end{aligned}
\tag{6.1}
$$

or $E_k^1 = A_1P_k^1, E_k^{k+1} = A_2P_k^{k+1}, \ldots$, where $A_m = (a_{ij}^m)_{i=\overline{1,k},j=\overline{1,i}}$, $E_k^1 = (e_1,\ldots,e_k)^T$, $E_k^{k+1} = (e_{k+1},\ldots,e_{2k})^T, \ldots, E_k^l = (e_l,\ldots,e_{l+k-1})^T$, the matrices $P_k^1, P_k^{k+1}, \ldots$ being defined in an analogous way.

The matrices $A_m$ are triangular and we may assume they have different dimensions. Relations (6.1) which define the set $\{e_n,\ n \in \mathbb{N}^*\}$ may be written in matrix form: $E = M \cdot P$, where $E = (E_k^1 E_k^{k+1} \ldots)^T, P = (P_k^1 P_k^{k+1} \ldots)^T$, and $M$ is a block diagonal infinite matrix: $\operatorname{diag}(M) = (A_1 A_2 \ldots)$.

Denoting by $G_n$ the Gram matrix associated to the blocks $A_1, A_2, \ldots, A_n$, we get:

$$\det(G_n - \lambda I_{n\cdot k}) = \begin{vmatrix} A_1 - \lambda I_k & & & \\ & A_2 - \lambda I_k & & \\ & & \ddots & \\ & & & A_n - \lambda I_k \end{vmatrix} = |A_1 - \lambda I_k| \cdot |A_2 - \lambda I_k| \cdot \ldots \cdot |A_n - \lambda I_k|.$$

In this way, the conditions $A < \infty$ and $B > 0$ are easily satisfied, taking for instance matrices $A_m$ having the same real positive eigenvalues.

The coefficients $a_{ij}^k$ must be taken such that $\overline{\operatorname{span}\{e_n, n \in \mathbb{N}^*\}} = \mathcal{H}$.

## 7. COMPARISON WITH EXISTING RESULTS

In [3] is given an equivalent condition under which a frame is a Riesz basis of a separable Hilbert space and there are obtained (using a different approach) formulas for the Riesz bounds: $\limsup_{n\to\infty} \frac{(\lambda_n^{\min})^2}{\lambda_n^{\max}}$ for the lower bound $B$ and $\liminf_{n\to\infty} \lambda_n^{\max}$ for the upper bound $A$. Using P4 it can be proved that $\frac{(\lambda_n^{\min})^2}{\lambda_n^{\max}}$ can be replaced by $\lambda_n^{\min}$ (which is a better value) and, at the same time, an optimal lower bound for the frame $\{x_i : i = \overline{1,n}\}$ considered in [3]. The conditions of closedness (condition 2° in Theorem 2) is imposed in [3] too.

The advantage of our results consists in the fact that the set $\{x_i : i \in \mathbb{N}\}$ need not be linearly independent, as it was assumed in [3]. Though, we considered here only real Hilbert spaces.

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*Technical University*
*of Cluj-Napoca*
*Department of Mathematics*
*str. Gh. Bariţiu 25*
*3400 Cluj-Napoca, Romania*
*E-mail:*
Daniela.Catinas@math.utcluj.ro