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DEGENERATED HOPF BIFURCATION IN THE FITZHUGH-NAGUMO SYSTEM. 2. BAUTIN BIFURCATION

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Abstract. The results on the Hopf bifurcation obtained in [1] are completed with those concerning the degenerated Hopf bifurcation of Bautin type. They are deduced by normal forms technique and concern a biodynamical system related with Van der Pol oscillator. Numerical investigation carried out by methods from [2] are also reported.

1. GOVERNING EQUATIONS

The Hopf bifurcation corresponds to the emergence of oscillatory regimes from equilibria and are of fundamental importance in science and engineering. Certain of the oscillations govern the heart functioning, whence the special interest in oscillator (e.g. Van der Pol) for biology and medicine. Among the most famous models used in these fields in [1] and [3] we considered the FitzHugh-Nagumo system

$$\dot{x} = c\left(x + y - x^3/3\right), \tag{1}$$

$$\dot{y} = -\left(x - a + by\right)/c, \tag{2}$$

where a, b and c are real parameters, $c > 2$. Suppose that c is fixed, therefore only two parameters (b and a) remain. For (1), (2) the Hopf bifurcation values (b, a) are situated on the Hopf bifurcation curves $H_{1,2}$ defined by the equations

$$a = \pm\frac{b}{3}\left(-2 + 3/b - b/c^2\right)\sqrt{1 - b/c^2},\ -c < b < c. \tag{3}$$

A Bautin bifurcation point is a degenerated Hopf bifurcation point at which the direction of the Hopf bifurcation is reversed: the subcritical Hopf bifurcation becomes a supercritical bifurcation or conversely. In addition the limit cycles existing subcritically, continue to exist supercritically too (and they have a very slow variation with the parameter). This phenomenon can occur in dynamical systems of dimension at least equal to two and for two or more parameters.

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At a Bautin bifurcation value, in the (b, a)-parameter space, a new curve Ba is formed near the Hopf bifurcation curve. It consists of those values at which two limit cycles collide and form a nonhyperbolic limit cycle. This paper is mainly aimed to determine this curve Ba and the value (b^*_{Ba}, a^*_{Ba}) at which it sets in from the Hopf bifurcation curve H_1. Due to the symmetry with respect to a of (1), (2) we shall find the Bautin bifurcation value and a curve of nonhyperbolic limit cycle values for $a < 0$, too.

Remind that the parameter c was assumed fixed and b^*_{Ba} and a^*_{Ba} were determined taking into account that (fixed) value of c.. If c varies, then, in the $(b,\ a)$ plane, (b^*_{Ba}, a^*_{Ba}) describes a curve, locus of Bautin bifurcation points. This curve will be also determined.

One among the methods used to prove the existence of the Hopf and Hopf degenerated bifurcation is based on the normal forms: the given system is transformed, by some homeomorphisms, into a typical simple (normal) form, topologically equivalent to the given system and presenting the Hopf bifurcation. The crucial point is that the first Liapunov coefficient is nonvanishing. If this condition is not satisfied, the Hopf bifurcation degenerates. Then, if the second Liapunov coefficient is nonvanishing, the Bautin bifurcation occurs [4], [5]. We recall that the Liapunov coefficients are the coefficients of the higher order terms in the normal form equation.

For an easier application of normal form theory we make the change of variables

$$x_1 = x - x^*, \quad x_2 = y - y^*. \tag{4}$$

Here $(x^*,\ y^*)$ is an equilibrium point which depends on a and b, namely x^* and y^* are the solutions of the equations

$$(x^*)^3 - 3x^*(1 - 1/b) - 3a/b = 0, \quad y^* = (a - x^*)/b. \tag{5}$$

In order to apply this theory, first we transform (1) and (2) such that $(x^*,\ y^*)$ be at the origin. Thus, replacing (4) into (1) and (2) and taking into account (5) we obtain the system

$$\dot{x}_1 = c\left[\left(1 - (x^*)^2\right)x_1 + x_2 - x^*x_1^2 - x_1^3/3\right], \quad \dot{x}_2 = -(x_1 + bx_2)/c, \tag{6}$$

which has the equilibrium point $(x_1,\ x_2) = (0,\ 0)$ for every pair of parameters $(b,\ a)$. Letting $\mathbf{x} = (x_1,\ x_2)^T$ and $\alpha = (b - b^*,\ a - a^*)$, the system (6) becomes

$$\dot{x} = \mathbf{A}x + \mathbf{F}(\alpha, x), \tag{7}$$

where the constant matrix $\mathbf{A}$ and the nonlinear vector $\mathbf{F}$ depend on the parameters and have the form

$$\mathbf{A} = \begin{pmatrix} c\left(1 - (x^*)^2\right) & c \\ -1/c & -(\alpha_1 + b^*)/c \end{pmatrix},$$

$$\mathbf{F} = \left(-cx^*x_1^2 - cx_1^3/3,\ 0\right)^T \equiv (F_1, F_2).$$

In this way we have $\mathbf{F}(\alpha, 0) = 0$, for every point (b^*, a^*) in the (b, a) plane. The matrix $\mathbf{A}$ has the eigenvalues $\lambda_{1,2} = \lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha) \equiv \mu + i\omega$, where

$$\begin{aligned} \mu(\alpha) &= \left[c^2\left(1 - (x^*)^2\right) - (\alpha_1 + b^*)\right]/(2c), \\ \omega(\alpha) &= \sqrt{4c^2 - \left[c^2\left(1 - (x^*)^2\right) + (\alpha_1 + b^*)\right]^2}/(2c) \end{aligned} \tag{8}$$

and x^* depends on α, a^* and b^*.

The point $(x^*,\ y^*)$ is a Hopf bifurcation point corresponding to the bifurcation value $\alpha^* = 0$ (i.e. $(b, a) = (b^*, a^*)$) if $\mu(0) = 0$, $\omega(0) \neq 0$ and $\ell_1(0) \neq 0$, where $\ell_1(\alpha)$ will be defined in the sequel.

Let us remark that $\mu(0) = 0$ implies

$$x^*(0) = \pm\sqrt{1 - b^*/c^2}, \tag{9}$$

hence the Hopf bifurcation values in the (b, a) plane are situated on the curves $H_{1,2}$ defined by (3), i.e.

$$a^* = \pm\frac{b^*\left(-2 + 3/b^* - b^*/c^2\right)}{3}\sqrt{1 - b^*/c^2},\ b^* \in (-c, c). \tag{10}$$

In this way, at the bifurcation value $(b, a) = (b^*, a^*)$ we have $\alpha = 0$ and $\omega^2(0) \equiv \omega_0^2 = 1 - (b^*)^2/c^2 > 0$.

For a simpler writing we introduce the notation

$$\begin{aligned} E_1 &\equiv E_1(\alpha) = c\left(1 - (x^*)^2\right), & E_{10} &\equiv E_1(0) = b^*/c, \\ E_2 &\equiv E_2(\alpha) = cx^2 & E_{20} &\equiv E_2(0) = \pm\sqrt{c^2 - b^*}, \\ E_3 &\equiv E_3(\alpha) = -b/c = -(\alpha_1 + b^*)/c, & E_{30} &\equiv E_3(0) = -b^*/c. \end{aligned} \tag{11}$$

With this notation, (6) and (8) become

$$\dot{x}_1 = E_1x_1 + cx_2 - E_2x_1^2 - cx_1^3/3, \quad \dot{x}_2 = -x_1/c + E_3x_2, \tag{12}$$

$$\mu = (E_1 + E_3)/2, \quad \omega = \sqrt{1 - (E_1 - E_3)^2/4}. \tag{13}$$

According to the values of the parameters occurring in E_1, E_2 and E_3, the system (12) can be reduced to various normal forms, each of which presenting a certain dynamic bifurcation phenomenon. For instance, the Hopf bifurcation occurs if (12) proves to be topologically equivalent to the normal form

$$\dot{\mathbf{z}} = (\beta_1 + i)\,\mathbf{z} + \beta_2 \mathbf{z}\,|\mathbf{z}|^2 + O(|\mathbf{z}|^4), \tag{14}$$

where $\beta_1(0) = 0$ and $\beta_1(0) = 0$.

The Bautin bifurcation takes place whenever, by means of several invertible transformations, the system (12) can be reduced to the normal form

$$\dot{\mathbf{z}} = (\beta_1 + i)\,\mathbf{z} + \beta_2 \mathbf{z}\,|\mathbf{z}|^2 + s\mathbf{z}\,|\mathbf{z}|^4 + O(|\mathbf{z}|^6), \tag{15}$$

where $\beta_1(0) = 0$ and $\beta_1(0) = 0$, $s = \pm 1$. In this way the phase portraits of (12) and (15) will be topologically equivalent. In (14) and (15), $\mathbf{z}$ is the new unknown function, $s = \pm 1$ and β_1 and β_2 are the new parameters depending on the former parameters α_1 and α_2. The bars stand for the modulus of the complex value of $\mathbf{z}$.

In order for this sequence of transformations be invertible some restrictions must be fulfilled [5]. Any time they occur, we shall specify them.

2. SYSTEMS EQUIVALENT TO (12)

The first transformation is defined with the aid of the eigenvector $\mathbf{q}$ of the matrix $\mathbf{A}$, which, with notation (11), reads $\mathbf{A} = \begin{pmatrix} E_1 & c \\ -1/c & E_3 \end{pmatrix}$, and the eigen vector $\mathbf{p}$ of $\mathbf{A}^T$. Here $\mathbf{q}$ corresponds to the eigenvalue λ_1 of $\mathbf{A}$ and $\mathbf{p}$ to $\overline{\lambda_1}$ and T stands for the transposition. Since $\mathbf{A}$ has real entries, its second eigenvalue is $\overline{\lambda_1}$. In general, $\mathbf{q}, \mathbf{p} \in \mathbb{C}^2$. In addition we have

$$\mathbf{q} = (q_1, (\lambda_1 - E_1)\,q_1/c)\,, \quad \mathbf{p} = (p1, c\,(E_1 - \overline{\lambda_1})\,p_1)\,, \tag{16}$$

where

$$\overline{p_1} q_1 = \left[1 - (\lambda_1 - E_1)^2\right]^{-1}. \tag{17}$$

It can be immediately checked that

$$\mathbf{A}\mathbf{q} = \lambda_1 \mathbf{q}, \ \mathbf{A}^T \mathbf{p} = \overline{\lambda_1}\mathbf{p}, \ \langle \mathbf{p}, \mathbf{q} \rangle = 1, \ \langle \mathbf{p}, \overline{\mathbf{q}} \rangle = 0. \tag{18}$$

Here $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{u_1} v_1 + \overline{u_2} v_2$ is the scalar product in $\mathbb{C}^2$. Since λ_1 is a complex number it follows that $\mathbf{p}, \mathbf{q} \in \mathbb{C}^2$. The bar over quantities stands for the complex conjugacy.

The first transformation reads

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) = \mathbf{z}\mathbf{q} + \overline{\mathbf{z}\mathbf{q}} \tag{19}$$

with $\mathbf{z}, \overline{\mathbf{z}} \in \mathbb{C}^2$. Taking the scalar product of the complexified form of (12) by $\mathbf{p}$ and taking into account $(18)_{3,4}$ we find, writing λ instead of λ_1

$$\dot{\mathbf{z}} = \lambda \mathbf{z} + g\,(\alpha, \mathbf{z}, \overline{\mathbf{z}})\,, \tag{20}$$

where

$$\begin{aligned} g\,(\alpha, \mathbf{z}, \overline{\mathbf{z}}) &= \langle p, F\,(\alpha, \mathbf{z}\mathbf{q} + \overline{\mathbf{z}\mathbf{q}}) \rangle = p_1 F_1\,(\alpha, \mathbf{z}\mathbf{q} + \overline{\mathbf{z}\mathbf{q}}) = \\ &= \sum_{k+l \geqslant 2} \tfrac{1}{k!l!} g_{kl}\,(\alpha)\,\mathbf{z}^k \overline{\mathbf{z}}^l = \tfrac{1}{2} g_{20} \mathbf{z}^2 + g_{11} \mathbf{z}\overline{\mathbf{z}} + \tfrac{1}{2} g_{02} \overline{\mathbf{z}}^2 + \\ &\quad + \tfrac{1}{6} g_{30} \mathbf{z}^3 + \tfrac{1}{2} g_{21} \mathbf{z}^2 \overline{\mathbf{z}} + \tfrac{1}{2} g_{12} \mathbf{z}\overline{\mathbf{z}}^2 + \tfrac{1}{6} g_{03} \overline{\mathbf{z}}^3. \end{aligned} \tag{21}$$

Equation (20) can be also obtained taking into account that $\mathbf{z} = \langle \mathbf{p}, \mathbf{x} \rangle$, hence $\dot{\mathbf{z}} = \langle \mathbf{p}, \dot{\mathbf{x}} \rangle$ where $\dot{\mathbf{x}}$ is given by (12). We found

$$\begin{aligned} g_{20} &= -2\overline{p}_1 E_2 q_1^2, & g_{30} &= -2c\overline{p}_1 q_1^3, \\ g_{11} &= -2\overline{p}_1 E_2 q_1 \overline{q}_1, & g_{21} &= -2c\overline{p}_1 q_1^2 \overline{q}_1, \\ g_{02} &= -2\overline{p}_1 E_2 \overline{q}_1^2, & g_{12} &= -2c\overline{p}_1 q_1 \overline{q}_1^2, \\ & & g_{03} &= -2c\overline{p}_1 \overline{q}_1^3. \end{aligned} \tag{22}$$

So far q_1 and p_1 satisfied relation (17) therefore one of them is still arbitrary. Then with the choice $q_1 = 1$ we find

$$\begin{aligned} g_{20} = g_{11} = g_{02} &= -2p_1 E_2 \equiv 2g, \\ g_{30} = g_{21} = g_{12} = g_{03} &= -2c\overline{p}_1 \equiv 6gr. \end{aligned} \tag{23}$$

and, correspondingly (21) becomes

$$\dot{\mathbf{z}} = \lambda \mathbf{z} + g\,(\mathbf{z} + \overline{\mathbf{z}})^2 + gr\,(\mathbf{z} + \overline{\mathbf{z}})^3. \tag{24}$$

Let us now perform another invertible transformation $(\mathbf{z}, \overline{\mathbf{z}}) \longleftrightarrow (\mathbf{w}, \overline{\mathbf{w}})$

$$\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 + \mathbf{z}_4 + \mathbf{z}_5 + O\left(\left|\mathbf{w}^6\right|\right), \tag{25}$$

where

$$\begin{aligned} \mathbf{z}_1 &= \mathbf{w}, \\ \mathbf{z}_2 &= \tfrac{h_{20}}{2} \mathbf{w}^2 + h_{11} \mathbf{w}\overline{\mathbf{w}} + \tfrac{h_{02}}{2} \overline{\mathbf{w}}^2, \\ \mathbf{z}_3 &= \tfrac{h_{30}}{6} \mathbf{w}^3 + \tfrac{h_{21}}{2} \mathbf{w}^2 \overline{\mathbf{w}} + \tfrac{h_{12}}{2} \mathbf{w}\overline{\mathbf{w}}^2 + \tfrac{h_{03}}{6} \overline{\mathbf{w}}^3, \\ \mathbf{z}_4 &= \tfrac{h_{40}}{24} \mathbf{w}^4 + \tfrac{h_{31}}{6} \mathbf{w}^3 \overline{\mathbf{w}} + \tfrac{h_{22}}{4} \mathbf{w}^2 \overline{\mathbf{w}}^2 + \tfrac{h_{13}}{6} \mathbf{w}\overline{\mathbf{w}}^3 + \tfrac{h_{04}}{24} \overline{\mathbf{w}}^4, \\ \mathbf{z}_5 &= \tfrac{h_{50}}{120} \mathbf{w}^5 + \tfrac{h_{41}}{24} \mathbf{w}^4 \overline{\mathbf{w}} + \tfrac{h_{32}}{12} \mathbf{w}^3 \overline{\mathbf{w}}^2 + \tfrac{h_{23}}{12} \mathbf{w}^2 \overline{\mathbf{w}}^3 + \\ &\quad + \tfrac{h_{14}}{24} \mathbf{w}\overline{\mathbf{w}}^4 + \tfrac{h_{05}}{120} \overline{\mathbf{w}}^5. \end{aligned} \tag{26}$$

The new unknown vector function is $\mathbf{w}$. Let us determine the so far arbitrary coefficients h_{ij} such that the differential equation for $\mathbf{w}$ becomes

$$\dot{\mathbf{w}} = \lambda \mathbf{w} + c_1 \mathbf{w}^2 \overline{\mathbf{w}} + c_2 \mathbf{w}^3 \overline{\mathbf{w}}^2. \tag{27}$$

Of course, (27) is exact up to $O\left(\left|\mathbf{w}^6\right|\right)$ as $\alpha \to 0$. Therefore these coefficients shall be determined from the relation

$$\dot{\mathbf{z}} = D\dot{\mathbf{w}} + K\dot{\overline{\mathbf{w}}} \tag{28}$$

where

$$\begin{aligned} D = 1 &+ h_{20} \mathbf{w} + h_{11} \overline{\mathbf{w}} + \tfrac{h_{30}}{2} \mathbf{w}^2 + \tfrac{h_{12}}{2} \overline{\mathbf{w}}^2 + \tfrac{h_{40}}{6} \mathbf{w}^3 + \tfrac{h_{31}}{2} \mathbf{w}^2 \overline{\mathbf{w}} + \\ &+ \tfrac{h_{22}}{2} \mathbf{w}\overline{\mathbf{w}}^2 + \tfrac{h_{13}}{6} \overline{\mathbf{w}}^3 + \tfrac{h_{50}}{24} \mathbf{w}^4 + \tfrac{h_{41}}{6} \mathbf{w}^3 \overline{\mathbf{w}} + \tfrac{h_{23}}{6} \mathbf{w}\overline{\mathbf{w}}^3 + \tfrac{h_{14}}{24} \overline{\mathbf{w}}^4, \end{aligned} \tag{29}$$

$$K = h_{11}\mathbf{w} + h_{02}\mathbf{w}\overline{\mathbf{w}} + \tfrac{h_{03}}{2}\overline{\mathbf{w}}^2 + \tfrac{h_{31}}{6}\mathbf{w}^3 + \tfrac{h_{22}}{2}\mathbf{w}^2\overline{\mathbf{w}} + \tfrac{h_{13}}{2}\mathbf{w}\overline{\mathbf{w}}^2 + \tfrac{h_{04}}{6}\overline{\mathbf{w}}^3 + \tfrac{h_{41}}{24}\mathbf{w}^4 + \tfrac{h_{23}}{4}\mathbf{w}^2\overline{\mathbf{w}}^2 + \tfrac{h_{14}}{6}\mathbf{w}\overline{\mathbf{w}}^3 + \tfrac{h_{05}}{24}\overline{\mathbf{w}}^4, \tag{30}$$

obtained by introducing (25) into (24). On the other hand, $\dot{\mathbf{w}}$ is given by (27). In this way (28) represents an identity of two polynomials in $\mathbf{w}$ and $\overline{\mathbf{w}}$, whence a system of affine algebraic equations in h_{ij} and c_1 and c_2.

With the exception corresponding to indices 21 and 32, all other equations allow the unique determination of h_{ij}. Indeed, each equation with $i+j=2$ contains a single h_{ij}; each equation with $i+j=3$, 4 or 5 contains a single h_{ij} with $i+j=3$, 4 or 5 respectively and also some previously determined h_{ij} with $i+j<3$, 4 or 5 respectively.

The equation for h_{21} contains, in addition, c_1 and the equation for h_{32} contains also c_2. It can be proved [5] that in order for the transition (25) be regular at $\alpha = 0$, it is necessary to choose $h_{32} = h_{21} = 0$. As a consequence, for the particular case of (6), these equations provide c_1 and c_2, namely

$$\begin{aligned} c_1 &= g\left(3r + 2\mathrm{Re}h_{11} + h_{20} + \overline{h}_{02}\right), \\ c_2 &= g\left(h_{12}/2\left(h_{20} + \overline{h}_{02} + 3r\right) + \mathrm{Re}h_{22} + r\left(h_{30} + \overline{h}_{03}\right)/2 + \right. \\ &\quad + \left(h_{31} + \overline{h}_{13}\right)/3 + 6r\left(\mathrm{Re}h_{11}^2 + |h_{11}|^2\right) + 6r\left(h_{20} + \overline{h}_{02}\right)\mathrm{Re}h_{11} + \\ &\quad + \left(h_{02} + \overline{h}_{20}\right)\left(h_{30} + \overline{h}_{03}\right)/6 + 3r/2\left|h_{20} + \overline{h}_{02}\right|^2 + \overline{h}_{12}\left(2\mathrm{Re}h_{11} + 3r\right). \end{aligned} \tag{31}$$

The final expressions of c_1 and c_2 in terms of g and r are given in [8]. They were obtained using the software MATHEMATICA [7]. Here we quote only

$$\begin{aligned} \mathrm{Re}c_1 = &\left[12\mu^3(\mathrm{Re}g)^2 - 4\mu^2\omega\,(\mathrm{Re}g)\,(\mathrm{Im}g) + 3\mu^4 r\,(\mathrm{Re}g) + \right. \\ &\left. +30\mu^2\omega^2 r\,(\mathrm{Re}g) + \mu G_1 + 9\omega^3 G_2 \mathrm{Re}g\right]/N, \end{aligned} \tag{32}$$

where

$$\begin{aligned} G_1 &= \omega^2\left[92\,(\mathrm{Re}g)^2 - 16(\mathrm{Im}g)^2\right], \\ G_2 &= 3r\omega - 4\mathrm{Im}g,\ N = \mu^4 + 10\mu^2\omega^2 + 9\omega^4. \end{aligned}$$

The third transformation concerns the independent variable. It reads

$$\tau = \omega\,(\alpha t) \tag{33}$$

and transforms (27) into

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} = (\nu(\alpha) + i)\,\mathbf{w} + d_1(\alpha)\,\mathbf{w}^2\overline{\mathbf{w}} + d_2(\alpha)\,\mathbf{w}^3\overline{\mathbf{w}}^2 + O\left(|\mathbf{w}|^6\right), \tag{34}$$

where

$$\nu(\alpha) = \mu(\alpha)/\omega(\alpha),\ d_1(\alpha) = c_1(\alpha)/\omega(\alpha),\ d_2(\alpha) = c_2(\alpha)/\omega(\alpha). \tag{35}$$

Here $\nu(\alpha)$ is a real function.

The next differential transformation introduces also a new time scale and has the form

$$\mathrm{d}\tau = \left\{1 - \mathrm{Im}d_1(\alpha)\,|\mathbf{w}|^2 + \left(\mathrm{Im}^2 d_1(\alpha) - \mathrm{Im}d_2(\alpha)\right)\mathbf{w}^4\right\}\mathrm{d}\theta. \tag{36}$$

It allows us to pass from (34) to

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\theta} = (\nu(\alpha) + i)\,\mathbf{w} + \ell_1(\alpha)\,\mathbf{w}^2\overline{\mathbf{w}} + \ell_2(\alpha)\,\mathbf{w}^3\overline{\mathbf{w}}^2 + O\left(|\mathbf{w}|^6\right). \tag{37}$$

The functions ℓ_1 and ℓ_2 are called the first and the second Liapunov coefficient respectively and they are given by

$$\begin{aligned} \ell_1(\alpha) &= \mathrm{Re}d_1(\alpha) - \nu(\alpha)\,\mathrm{Im}d_1(\alpha), \\ \ell_2(\alpha) &= \mathrm{Re}d_2(\alpha) - \mathrm{Re}d_1(\alpha)\,\mathrm{Im}d_1(\alpha) + \nu(\alpha)\left[\mathrm{Im}^2 d_1(\alpha) - \mathrm{Im}d_2(\alpha)\right]. \end{aligned}$$

3. THE LOCUS OF THE BAUTIN BIFURCATION VALUES

In the FitzHugh-Nagumo case we have, for $\alpha = 0$

$$\begin{aligned} \omega_0 &= \sqrt{\left(1 - (b^*)^2/c^2\right)},\ \overline{p}_1 = 1/2 - ib^*/(2c\omega_0), \\ g &= -\sqrt{c^2 - b^*}/2 + ib^*\sqrt{c^2 - b^*}/(2c\omega_0),\ r = c/\left(3\sqrt{c^2 - b^*}\right), \\ \ell_1(0) &= \mathrm{Re}c_1(0)/\omega_0 = \mathrm{Re}g\,(3\omega_0 r - 4\mathrm{Im}g)/\omega_0^2 = \\ &= \left(-(b^*)^2 + 2b^*c^2 - c^2\right)/(2\omega_0^3 c). \end{aligned}$$

For those b_H^* for which $\ell_1(0) \neq 0$ the normal form (37) presents a Hopf bifurcation around the points $P_H^* = \left(x_H^*(0, b_H^*, a_H^*),\ y_H^*(0, b_H^*, a_H^*),\ b_H^*,\ a_H^*\right)$. For those values b_{Ba}^* for which $\ell_1(0) = 0$, i.e.

$$(b_{Ba}^*)^2 - 2b_{Ba}^* c^2 + c^2 = 0, \tag{38}$$

the Hopf bifurcation fails and a Bautin bifurcation takes place around the point $P_{Ba}^* = \left(x_{Ba}^*(0, b_{Ba}^*, a_{Ba}^*),\ y_{Ba}^*(0, b_{Ba}^*, a_{Ba}^*),\ b_{Ba}^*,\ a_{Ba}^*\right)$, where (b_{Ba}^*, a_{Ba}^*) are also situated on the Hopf bifurcation curves. Obviously we have

$$b_{Ba}^* = c^2 - c\sqrt{c^2 - 1}. \tag{39}$$

Taking into account (3), it follows the corresponding a_{Ba}^* values

$$a_{Ba}^* = \pm\tfrac{4}{3}\left(c\sqrt{c^2 - 1} - c^2 + 1\right)\sqrt[4]{1 - 1/c^2}. \tag{40}$$

Then, from (9) we obtain

$$x_{Ba}^* = \pm\sqrt[4]{1 - 1/c^2}. \tag{41}$$

Elimination of c between (39) and (40) yields in the (b,a)-plane the curve locus of Bautin bifurcation points. In order to determine it, let us recall that our reasoning made in the quoted previous papers are valid for $c>2$, in which case $b^*>0$. Since taking into account (39) and (40) we have

$$a^*_{Ba} = \pm\tfrac{4}{3}\left(1-b^*_{Ba}\right)\sqrt[4]{1-1/c^2}. \tag{42}$$

It follows

$$\left(3a^*_{Ba}/\left[4\left(1-b^*_{Ba}\right)\right]\right)^2 = \left(1-b^*_{Ba}\right)/b^*_{Ba}, \tag{43}$$

whence $b^*_{Ba}\leqslant 1$. Then from (39) we see that $b^*_{Ba}\longrightarrow 1/2$ as $c\longrightarrow\infty$. Correspondingly $a^*_{Ba}\in\left(-\frac{2}{3},\frac{2}{3}\right)$. Finally (43) reads

$$a^*_{Ba} = \pm\tfrac{4}{3}\left(1-b^*_{Ba}\right)\sqrt{1/b^*_{Ba}-1},\ b^*_{Ba}\in(0.5,1) \tag{44}$$

and represents the curve locus of the Bautin bifurcation values (Figure 1).

This curve has a limit point corresponding to $b^*_{Ba}=1/2$. This explains why the experiments revealed no Bautin phenomenon for $b^*_{Ba}<1/2$.

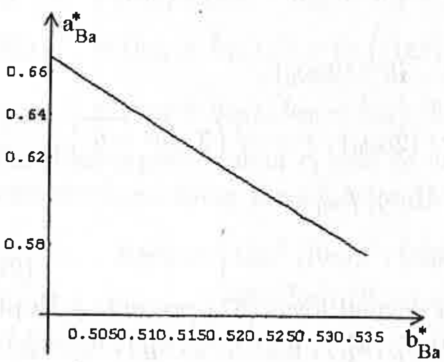


Fig. 1. The curve locus of the Bautin bifurcation point for $a>0$.

Our numerical experiments were carried out at $c=5$ and b fixed, namely $b=0.5;\ 0.6;\ 0.7;\ 0.8;\ 0.9;\ 1;\ 1.2;\ 1.3$ and a variable.

They showed the existence of a Bautin bifurcation value at about

$$\left(b^*_{Ba\,exp}, a^*_{Ba\,exp}\right) = (0.5,\ 0.65).$$

They agree very well with the theoretical values given by (39) and (40) for $c=5$ and $a>0$, namely $b^*_{Ba}=25-5\sqrt{24}\approx 0.505103$ and

$$a^*_{Ba} = \tfrac{4}{3}\left(5\sqrt{24}-24\right)\sqrt[4]{24/25}\approx 0.653163.$$

Let us recall that the Bogdanov-Takens bifurcation values [4], [5] are situated at the intersection of the curves $S_{1,2}$ (corresponding to double equilibrium points) and the Hopf bifurcation curves $H_{1,2}$. Hence, these values $\left(b^*_{BT}, a^*_{BT}\right)$

describe, as c is varied, part of the curves $S_{1,2}$. Let us call them $S_{1,2\,BT}$; they are locus of the Bogdanov-Takens bifurcation values.

Unlike this situation, the locus of the Bautin bifurcation values (44) is not related to $S_{1,2}$. For each c fixed, the Bogdanov-Takens bifurcation values are limit points of the Hopf bifurcation curves $H_{1,2}$ whereas the Bautin bifurcation values belong to $H_{1,2}$. In fact, the genuine locus of the Hopf bifurcation value are $H_{1,2}$ except for $\left(b^*_{Ba}, a^*_{Ba}\right)$ i.e. it consists of the curves (3) defined for

$$b\in\left(-c, b^*_{Ba}\right)\cup\left(b^*_{Ba}, c\right) \tag{45}$$

In this way the Bautin bifurcation values are also limit points for the Hopf bifurcation curves described by the equations (3) for the domain (45) of b.. However, since the Bogdanov-Takens as well as Bautin bifurcation values are as close as wanted to Hopf bifurcation point, the Bautin and Bogdanov-Takens bifurcations are Hopf degenerated bifurcations.

The pair $\left(b^*_{Ba}, a^*_{Ba}\right)$ is a Bautin bifurcation point if the corresponding second Liapunov coefficient has the property $\ell_2(0)\neq 0$. Since $\ell_1(0)=0$, $\nu(0)=0$, from (35) we have $\ell_2(0)=\mathrm{Re}c_2(0)/\omega_0$. Taking into account the expression of $\mathrm{Re}c_2$ from [8] and using the above expressions for ω_0, g and r in which b^*_{Ba} is given by (39) we find

$$\ell_2(0) = \frac{40b^*_{Ba}}{9\omega_0^4}\mathrm{Re}g\mathrm{Im}g\neq 0. \tag{46}$$

Hence, all points of the curve (44) are Bautin bifurcation points.

4. THE NONHYPERBOLIC LIMIT CYCLE BIFURCATION

In order to pass from (37) to its normal form (15) we need an invertible parameter transformation

$$\mu_1=\nu(\alpha),\ \mu_2=\ell_1(\alpha), \tag{47}$$

which is regular at $\alpha=0$, i.e.

$$\det\begin{pmatrix}\frac{\partial\nu}{\partial\alpha_1} & \frac{\partial\nu}{\partial\alpha_2}\\ \frac{\partial\ell_1}{\partial\alpha_1} & \frac{\partial\ell_1}{\partial\alpha_2}\end{pmatrix}_{|\alpha=0} = \frac{1}{\omega_0}\det\begin{pmatrix}\frac{\partial\mu}{\partial\alpha_1} & \frac{\partial\mu}{\partial\alpha_2}\\ \frac{\partial\ell_1}{\partial\alpha_1} & \frac{\partial\ell_1}{\partial\alpha_2}\end{pmatrix}_{|\alpha=0}\neq 0. \tag{48}$$

Thus (37) becomes

$$\dot{\mathbf{w}} = (\mu_1+i)\mathbf{w}+\mu_2\mathbf{w}|\mathbf{w}|^2+L_2(\mu)\mathbf{w}|\mathbf{w}|^4+O\left(|\mathbf{w}|^6\right), \tag{49}$$

where $L_2(\mu)=\ell_2(\alpha(\mu))$ is a smooth function of μ with $L_2(0)=\ell_2(0)\neq 0$. Finally we use the rescaling [5]

$$\mathbf{w}=\sqrt[4]{|L_2(\mu)|}\mathbf{u},\ \mathbf{u}\in\mathbb{C}^2 \tag{50}$$

and introduce other new parameters

$$\beta_1 = \mu_1, \ \beta_2 = \sqrt{|L_2(\mu)|}\mu_2 \tag{51}$$

to produce equation (15), where instead of $\mathbf{u}$ we wrote $\mathbf{z}$, presenting the Bautin bifurcation phenomenon at $(\mathbf{u}, \beta_1, \beta_2) = (\mathbf{0}, 0, 0)$. In (15) we have $s = \mathrm{sign}\ell_2(0)$. In these conditions, in the (β_1, β_2)-plane there exists a curve Ba tangent to H_1 at $Q(b^*_{Ba}, a^*_{Ba})$, such that for points of Ba the corresponding equation (15) possesses two colliding limit cycles (that is a nonhyperbolic limit cycle). It reads

$$\beta_2^2 + 4\beta_1 = 0, \ \beta_2 > 0. \tag{52}$$

Coming back to our initial variables, the equation of this curve has the form $\ell_2\ell_1^2 + 4\nu = 0$ or equivalently

$$\begin{aligned}&\left[\mathrm{Rec}_2 - \tfrac{1}{\omega}\mathrm{Rec}_1\mathrm{Imc}_1 + \mu\left(\mathrm{Imc}_1\right)^2 - \mathrm{Imc}_2\right] \cdot \\ &\qquad \cdot \left(\tfrac{1}{\omega}\mathrm{Rec}_1 - \tfrac{\nu}{\omega}\mathrm{Imc}_1\right)^2 + 4\mu = 0.\end{aligned} \tag{53}$$

The involved equations depend on $\alpha = (b - b^*_{Ba}, \ a - a^*_{Ba})$.

For our concrete equation (12) we represented (53) in Figure 2 for the case $c = 5$. Since Ba is very close to H_1, in Figure 2 we gave a qualitative representation. This was supplemented with Table 1 for values (b, a) situated on Ba and H_1. The first three decimals of a_{Ba} obtained theoretically are the same as those obtained numerically.

Table 1.

Values (b, a) situated on the Hopf curve H_1 and on the curve Ba

b	a_{H_1}	a_{Ba}
0.55	0.6223	0.6227
0.60	0.5880	0.5889
0.70	0.5193	0.521
0.80	0.4507	0.453
0.90	0.3821	0.386
1.00	0.3135	0.318
1.20	0.1764	0.183
1.30	0.1078	0.115
1.3104	0.1034	0.108

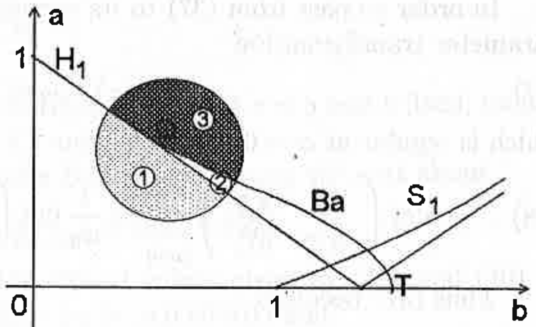
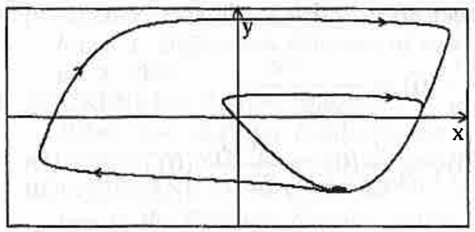
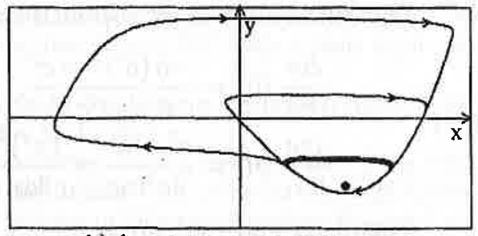


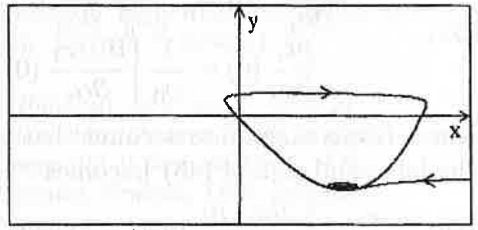
Fig. 2. The curve Ba of nonhyperbolic limit cycle bifurcation values and the point Q of Bautin bifurcation, for $c = 5$.



a) $b = 0.55, a = 0.6$



b) $b = 0.55, a = 0.6227481$



c) $b = 0.55, a = 0.63$

Fig. 3. Phase portraits for $c = 5$ and (b, a) in domains 1, 2, 3 from Figure 2.

In Figure 3, phase portraits for the different regions of Figure 2 around Q are represented. In Figure 3a, the trajectory through $(1, 0.2)$ is drawn for parameters (b, a) in domain 1 of Figure 2. Its α-limit point is the only equilibrium point (repulsor) and its ω-limit set is the attractive limit cycle. In Figure 3b, the trajectory through the same point is represented for (b, a) in domain 2 of Figure 2. Its α-limit set is the repulsive limit cycle and its ω-limit set is the attractive limit cycle. The only equilibrium point is an attractor, situated inside the repulsive limit cycle. In Figure 3c, the trajectory through the point $(1, 0.2)$ is represented for parameters in domain 3 of Figure 2. It emerges from infinity, while its ω-limit set is the only equilibrium point, the attractor. Both limit cycles disappeared, due to their collision that took place for (b, a) situated on the curve Ba.

It remains to prove (48) for the concrete case of (12). Thus, taking into account (13) it follows

$$\overline{p}_1 = 1/2 + \mathrm{i}\left(\nu/2 - c\left(1 - (x^*)^2\right)/(2\omega)\right) \tag{54}$$

whence

$$\begin{aligned} \mathrm{Re}g &= -cx^*/2, \\ \mathrm{Im}g &= E_2\left(E_1 - E_3\right)/(4\omega) = -cx^*/2 + c^2x^*\left(1 - (x^*)^2\right)/(2\omega), \\ r &= 1/(3x^*), \ [r\mathrm{Re}g](0) = -c/6, \ [\mathrm{Im}g](0) = 3\omega_0 r(0)/4. \end{aligned} \tag{55}$$

The differentiation of (5) with respect to a and b leads to

$$\begin{aligned}&\frac{\partial\nu}{\partial\alpha_1}(0)=\frac{-5\,(b^*)^2-c^2}{6c\omega_0\,(b^*)^2},\ \frac{\partial\nu}{\partial\alpha_2}(0)=\frac{-c}{2\omega_0 b^* x^*(0)},\\ &\frac{\partial\omega}{\partial\alpha_1}(0)=\frac{c^2-b^*}{3c^2\omega_0}=\frac{(x^*)^2}{3\omega_0}(0),\ \frac{\partial\omega}{\partial\alpha_2}(0)=\frac{-c^2x^*}{2\omega_0 b^*}(0).\end{aligned}\tag{56}$$

Then (32) and (35) imply

$$\begin{aligned}&\frac{\partial \mathrm{Re}c_1}{\partial\alpha_i}(0)=\frac{G_1(0)}{9\omega_0^2}\frac{\partial\nu}{\partial\alpha_i}(0)-\frac{cx^*}{2\omega_0}(0)\frac{\partial\,(3\omega r-4\mathrm{Im}g)}{\partial\alpha_i}(0),\\ &\frac{\partial\ell_1}{\partial\alpha_i}(0)=\frac{1}{\omega_0}\left[\frac{\partial\mathrm{Re}c_1}{\partial\alpha_i}(0)-\mathrm{Im}c_1(0)\frac{\partial\nu}{\partial\alpha_i}(0)\right],\end{aligned}\tag{57}$$

where it was taken into account that $\mathrm{Re}c_1(0)=0$ and $\mu(0)=\nu(0)=0$. Thus, the left hand side of (48) becomes

$$\left[\frac{\partial\nu}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2}(3\omega r-4\mathrm{Im}g)-\frac{\partial\nu}{\partial\alpha_2}\frac{\partial}{\partial\alpha_1}(3\omega r-4\mathrm{Im}g)\right]_0.\tag{58}$$

Here $3\omega r-4\mathrm{Im}g=2cx^*\nu+\dfrac{\omega^2-2c^2(x^*)^2\left(1-(x^*)^2\right)}{x^*\omega}$, therefore we must consider only

$$\frac{\partial}{\partial\alpha_i}(3\omega r-4\mathrm{Im}g)(0)=\left\{\frac{\partial}{\partial\alpha_i}\left[\frac{\omega^2-2c^2(x^*)^2\left(1-(x^*)^2\right)}{\omega x^*}\right]\right\}_0\tag{59}$$

In this way (58) reads

$$\begin{aligned}&\frac{2}{x^2}\left(\frac{\partial\nu}{\partial\alpha_1}\frac{\partial\omega}{\partial\alpha_2}-\frac{\partial\nu}{\partial\alpha_2}\frac{\partial\omega}{\partial\alpha_1}\right)-\\ &-\frac{\omega_0^2-2c^2(x^*)^2\left(1-3(x^*)^2\right)}{\omega_0(x^*)^2}\left[\frac{\partial\nu}{\partial\alpha_1}\frac{\partial x^*}{\partial\alpha_2}-\frac{\partial\nu}{\partial\alpha_2}\frac{\partial x^*}{\partial\alpha_1}\right]=\\ &=\frac{2}{x^2}\frac{cx^*\left(c^2+3b^*\right)}{6\omega_0^2(b^*)^2}+\frac{4c^2(x^*)^2}{\omega_0}\frac{1}{4c\omega_0 b^*(x^*)^2}=\frac{c^3}{3\omega_0^2 b^*}\neq 0,\end{aligned}\tag{60}$$

which proves (48).

R E F E R E N C E S

[1] GEORGESCU, A., ROCŞOREANU, C., GIURGIŢEANU, N., *Equilibria and relaxation oscillations of the nodal system of the heart. 2. Hopf bifurcation,* Rev. Roum. Sci. Tech. – Mec. Appl. (in press).

[2] GIURGIŢEANU, N., *Study of the phase portrait. I. Improvements of the numerical methods,* St. Cerc. Mec. Apl., **55**, nos. 5–6, pp. 335–355, 1996 (in Romanian).

[3] ROCŞOREANU, C., *Equilibria and relaxation oscillations of the nodal system of the heart. 1. Bifurcation diagrams of equilibria,* Rev. Roum. Sci. Tech. – Mec. Appl., **43**, no. 3, 1998.

[4] GUCKENHEIMER, J., HOLMES, P., *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields,* Springer, New York, 1983.

[5] KUZNETSOV, YU., *Elements of Applied Bifurcation Theory,* Springer, New York, 1995.

[6] ROCŞOREANU, C., GIURGIŢEANU, N., GEORGESCU, A., *Degenerated Hopf bifurcation in the FitzHugh-Nagumo system. 1. Bogdanov-Takens bifurcation,* Ann. of Univ. of Timişoara, **XXXV**, no. 2, pp. 285–298, 1997.

[7] *** MATHEMATICA, A system for doing mathematics by computer, 1994.

[8] ROCŞOREANU, C., *Dynamics and bifurcation in the Fitz-Hugh-Nagumo equation,* Romanian Academy, Bucharest, PhD Thesis, 1997 (in Romanian).

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