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ON A CLASS OF DIMSIMs METHODS

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Abstract. We provide conditions concerning the partitioned block matrix from the Runge-Kutta methods in order to obtain methods of given order.

1. INTRODUCTION

The general linear methods for the numerical solutions of the initial value problem

$$y' = f(x, y(x)) \tag{1.1}$$

$$y(x_0) = y_0 \tag{1.2}$$

with given $f : [a,b] \times R^m \to R^m$, $y_0 = a$, $y_0 \in R^m$, introduced by J. C. Butcher, [1], have been discussed by many authors, [1]–[7]. This type of numerical methods includes, as particular cases, many known methods: linear multistep methods, Runge-Kutta methods, predictor-corrector methods, etc. The general linear methods with s stages are defined by:

$$Y_i^{[n+1]} = h \sum_{j=1}^{s} a_{ij} f(x_n + c_j h,\ Y_j^{[n+1]}) + \sum_{j=1}^{r} u_{ij} y_j^{[n]};\ i = 1, 2, ..., s, \tag{1.3}$$

$$y_i^{[n+1]} = h \sum_{j=1}^{s} b_{ij} f(x_n + c_j h,\ Y_j^{[n+1]}) + \sum_{j=1}^{r} v_{ij} y_j^{[n]};\ i = 1, 2, ..., r, \tag{1.4}$$

where $n = 0, 1, 2, ..., N-1$, $h = (b-a)/N$, $x_n = x_0 + nh$, $n = 0, 1, 2, ..., N$.

The methods given by (1.3)–(1.4) can be represented as a partitioned $(s+r) \times (s+r)$ matrix

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(1.5) $$\begin{pmatrix} A & U \\ B & V \end{pmatrix},$$

with

$$A = (a_{ij});\ B = (b_{ij}),\ U = (u_{ij}),\ V = (v_{ij}).$$

If the matrix A is strictly lower triangular, then the general linear methods (1.3)–(1.4) are called explicit and in opposite case, implicit. The diagonally implicit multistage integration methods (DIMSIMs), investigated by Butcher and Jackiewicz, [2], [3], have a matrix A lower triangular, with a constant value on the diagonal. This class of methods can be divided into four types, [3], depending on the structure of the matrix A.

The vectors $Y_i^{[n+1]}$, $i = \overline{1, s}$, called internal stages of the method, give approximations of order q (the stage order) for the solution of the problem (1.1)–(1.2) at points $x_n + c_i h$, $i = \overline{1, r}$, $y_i^{[n+1]}$ called internal stages, give approximations of order p (the order of method), of the solution of the problem (1.1)–(1.2) at the point $x_{n+1} = x_n + h$. We recall that the general linear methods (1.3)–(1.4) have the stage order q and the order p, if there exist vectors $\alpha_k = (\alpha_{1k}, \alpha_{2k}, ..., \alpha_{nk})^{\mathrm{T}}$, $k = 0, 1, 2, ..., p$, such that under the assumption

(1.6) $$y_i^{[n]} = \sum_{k=0}^{p} \alpha_{ik} y^{(k)}(x_n)\, h^k + O\left(h^{p+1}\right),$$

one has

(1.7) $$Y_i^{[n]} = y(x_n + c_i h) + O(h^{q+1}),\ i = 1, 2, ..., s,$$

(1.8) $$y^{[n+1]} = \sum_{k=0}^{p} \alpha_k y^{(k)}(x_n + h) + O(h^{p+1}).$$

The matrix V determines the stability of the method (1.3)–(1.4). The method is stable (or zero stable) if V is power bounded, i. e.

(1.9) $$\sup_n \|V^n\| < \infty.$$

The aim of this paper is to highlight a class of DIMSIMs of the type 1, with $p = q = r = s$, in section 2, by an extension of a result of J. C. Butcher ([3], Theorem 5.1). Also we derive a family of such stable methods depending on two parameters, in section 3.

2. ORDER CONDITIONS FOR A NEW CLASS OF DIMSIMs

First, we recall one result of Butcher, from [3], expressed by

THEOREM 2.1. *(Theorem 3.1 from [3]). The method (1.3)–(1.4) has order $p = q$, if the matrices A, B, U, V satisfy*

(2.1) $$e^{cz} = zAe^{cz} + Uw + O(z^{p+1}),$$

(2.2) $$e^{z} \cdot w = zBe^{cz} + Vw + O(z^{p+1}),$$

where $c = (c_1, c_2, ..., c_s)^{\mathrm{T}}$, $e^{cz} = (e^{c_1 z}, e^{c_2 z}, ..., e^{c_s z})^{\mathrm{T}}$ and $w = (w_1, w_2, ..., w_r)$ is the vector with components

$$w_i = \sum_{k=0}^{p} \alpha_{ik} z^k,\ i = 1, 2, ..., r.$$

Now, we will extend the conclusion of the theorem 5.1, [3], valid only if the matrix U is the identity matrix, by the next theorem

THEOREM 2.2. *The DIMSIM*

(2.3) $$\begin{pmatrix} A & U \\ B & V \end{pmatrix},$$

has the order $p = q = r = s$, if and only if, the matrices A, B, U, V satisfy

(2.4) $$UV = VU$$

and

(2.5) $$UB = B_0 - AB_1 - VB_2 + VA,$$

where B_0, B_1, B_2 are matrices of order s having respectively the elements

(2.6) $$b_{ij}^0 = \frac{\int_0^{1+c_j} \varphi_j(x)\,dx}{\varphi_j(c_j)},\ b_{ij}^1 = \frac{\varphi_j(1+c_i)}{\varphi_j(c_j)},\ b_{ij}^2 = \frac{\int_0^{c_i} \varphi_j(x)\,dx}{\varphi_j(c_j)}$$

and $\varphi_j(x) = \overline{1, s}$ is the polynomial

$$\varphi_j(x) = \prod_{k=1,\, k\neq j}^{s} (x - c_k).$$

Proof. Using Theorem 2.1 and (2.4), we obtain

(2.7) $$Uw = (I - zA)e^{cz} + O(z^{p+1}),$$

(2.8) $$e^z Uw = zUBe^{cz} + VUw + O(z^{p+1}),$$

with I – the identity matrix.

From (2.7) and (2.8) we obtain

$$VUw = Ve^{cz} - VAze^{cz} + O(z^{p+1}),$$

$$(UB - VA)ze^{cz} = e^z \cdot e^{cz} - A(ze^z e^{cz}) - Ve^{cz} + O(z^{p+1}). \tag{2.9}$$

For $z = 0$, we see that the condition $Ve = e = (1, 1, ..., 1)^{\mathrm{T}} \in R^s$, hold. To finish the proof, with (2.9) we follow exactly the Butcher's proof, [3], and obtain the conclusion (2.5). □

Remark 2.1. The order condition (2.5) from Theorem 2.2, offers the possibility to construct a class of DIMSIMs of order p, which includes the general class found by Butcher, [3], p. 356.

3. A FAMILY OF STABLE DIMSIMs OF ORDER 2

In the following, using Theorem 2.2, we will derive a family of stable DIMSIMs with $p = q = r = s = 2$ of type 1, depending on two parameters. For this, we select the matrices A, B, U, V of the form

$$A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{3.1}$$
$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad V = \begin{pmatrix} v & 1-v \\ v & 1-v \end{pmatrix}$$

satisfying (2.4), (2.5).

We take $c = (c_1, c_2)^{\mathrm{T}} = (0, 1)^{\mathrm{T}}$ and then the matrices B_0, B_1, B_2 from (2.5) are

$$B_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{3.2}$$

Remark 3.1. In order to apply Theorem 2.2 we need matrices U, V such that

$$UV = VU.$$

If we take

$$U = \begin{pmatrix} \alpha & \beta \\ \beta\gamma & \alpha + \beta - \beta\gamma \end{pmatrix}, \; V = \begin{pmatrix} \frac{\gamma}{1+\gamma} & \frac{1}{1+\gamma} \\ \frac{\gamma}{1+\gamma} & \frac{1}{1+\gamma} \end{pmatrix}, \tag{3.3}$$

with α, β, $\gamma \neq -1$ real parameters, then (2.4) hold, and for $\alpha = 0$, $\beta = 0$ follows that $U = I$, where I is identity matrix.

The proof follows immediately. In this case, the order condition (2.5) can be written as the system

$$\begin{aligned} \alpha b_{11} + \beta b_{21} &= \frac{2a-1}{2(1+\gamma)} + \frac{1}{2}, \\ \beta\gamma b_{11} + (\alpha + \beta - \beta\gamma) b_{21} &= \frac{2a-1}{2(1+\gamma)}, \\ \alpha b_{12} + \beta b_{22} &= \frac{\gamma}{2(1+\gamma)}, \\ \beta\gamma b_{12} + (\alpha + \beta - \beta\gamma) b_{22} &= 2 - a - \frac{1}{2(1+\gamma)}. \end{aligned} \tag{3.4}$$

From the system (3.4) follows

$$\begin{aligned} b_{11} &= \frac{(2\alpha+\gamma)(\alpha - \beta\gamma) + \beta(1+\gamma)}{2(\alpha+\beta)(1+\gamma)(\alpha - \beta\gamma)}, \\ b_{12} &= \frac{\gamma(\alpha - \beta\gamma) + (2a-3)\beta(1+\gamma)}{2(\alpha+\beta)(1+\gamma)(\alpha - \beta\gamma)}, \\ b_{21} &= \frac{(2a-1)\alpha - \beta\gamma(2a+\gamma)}{2(\alpha+\beta)(1+\gamma)(\alpha - \beta\gamma)}, \\ b_{22} &= \frac{\gamma(\alpha - \beta\gamma) - (2a-3)\alpha(1+\gamma)}{2(\alpha+\beta)(1+\gamma)(\alpha - \beta\gamma)} \end{aligned} \tag{3.5}$$

with α, β, γ free parameters, subject to $\alpha \neq -\beta$, $\gamma \neq -1$, $\alpha \neq \beta\gamma$.

The remaining parameters α, β, γ will be obtained from the condition

$$\det M(z) = 0. \tag{3.6}$$

Here, the stability matrix $M(z)$, is given by

$$M(z) = V + zB(I - zA)^{-1}U, \tag{3.7}$$

with A, B from (3.1) and U, V from (3.3).

After a tedious computation one has

$$\det M(z) = \frac{A}{N(1+\gamma)} z + \frac{1}{N}\left(\frac{B}{1+\gamma} + \frac{C}{N}\right) z^2, \tag{3.8}$$

where

$$N = 2(\alpha+\beta)(1+\gamma)(\alpha - \beta\gamma),$$

$$A = \frac{N}{2}(1 - 2a - 3\gamma), \; B = \frac{1}{2} N a (2a - 3), \tag{3.9}$$

$$C = \frac{N^2}{2(1+\gamma)}((2-a)\gamma - a(2a-3))$$

By substituting (3.9) in (3.8) we obtain

$$\det M(z) = \frac{1}{2(1+\gamma)}\left\{(1-(2a-3)\gamma)z + (2-a)\gamma\cdot z^2\right\}. \tag{3.10}$$

We see, that for $a = 2$ and $\gamma = 1$ follows

$$\det M(z) = 0.$$

So, for $a = 2$, $\gamma = 1$, with (3.5) we have obtained one family of DIMSIMs of type 1 and of order $p = 2 = q$ depending on two parameters. This family is characterized by the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} \frac{5\alpha-3\beta}{4(\alpha^2-\beta^2)} & \frac{1}{4(\alpha-\beta)} \\ \frac{3\alpha-5\beta}{4(\alpha^2-\beta^2)} & \frac{-1}{4(\alpha-\beta)} \end{pmatrix}, \quad U = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, V = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \tag{3.11}$$

where $\alpha, \beta \in R$, $\alpha \neq \beta$, $\alpha \neq -\beta$, are free parameters.

Remark 3.2. Because $\|V^n\| = 1$, $n \in N^*$, the matrices (3.11) provide stable methods.

Remark 3.3. If we select for parameters α, β the values $\alpha = 1$, $\beta = 0$ we obtain the DIMSIM given by the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} 5/4 & 1/4 \\ 3/4 & -1/4 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}. \tag{3.12}$$

This particular DIMSIM has been found by J. C. Butcher, [3], section 6, using as matrix U the identity matrix I.

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