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THE ACCELERATION OF THE CONVERGENCE OF CERTAIN APPROXIMANT SEQUENCES OF THE SOLUTIONS OF CERTAIN EQUATIONS IN NORMED LINEAR SPACES

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Abstract. We develop some ideas from [2] and [3]: given a mapping between two linear spaces, a corresponding equation and an iterative process for approximating a solution of this equation, we study some techniques of improving the convergence of the iterative process. This will be accomplished by considering another sequence, depending on the first one, but which does not require additional essential operations. The importance of our results is illustrated by a particular case.

In this paper we will develop the ideas presented in the papers [1], [2], [3]. As in the aforementioned papers, let us consider the linear normed spaces X and Y, let us note their norms with $\|.\|_X$ and $\|.\|_Y$ respectively, while θ_Y is the null element of the space Y. Then, the set $D \subseteq X$ and the function $f : D \to Y$ being given, the request is to determine an element $x^* \in D$ which will be the solution of the equation:

$$f(x) = \theta_Y. \tag{1}$$

In order to determine this element we will use $(x_n)_{n\in\mathbb{N}} \subseteq D$ approximant sequences. As the n rank term of this sequence is x_n, the error with which this term approximates the solution x^* of the equation (1) being $\|x^* - x_n\|_X$, we wish to make this error decrease as n increases, and, especially, being given a number $\varepsilon > 0$ which we call the maximum admissible error, to determine that rank n starting from which the inequality:

$$\|x^* - x_n\|_X \leqslant \varepsilon \tag{2}$$

is verified.

Then we want the inequality (2) to be valid for values of the number $n \in \mathbb{N}$ that are as small as possible. Nothing that the determination of a new

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approximation x_{n+1} is based on a recurrence relation according to the known anterior approximations, such a determination will be called iteration step. Thus, the accomplishing of the above mentioned desideratum means to reach the maximum admissible error after as few iteration steps as possible, that is to achieve a good convergence speed, a speed given by the notion of order of an approximant sequence, this notion being introduced in [2], through the following definition.

DEFINITION 1. *Let us consider the above elements, the number $p \in \mathbb{N}$ not null and $(x_n)_{n\in\mathbb{N}} \subseteq D$, an approximant sequence of a solution of the equation (1).*

We say that the approximant sequence $(x_n)_{n\in\mathbb{N}}$ is of the order p if $\alpha, \beta \geqslant 0$ exist so that for any $n \in \mathbb{N}$ we have:

$$\begin{cases} \|f(x_{n+1})\|_Y \leqslant \alpha \|f(x_n)\|_Y^p ; \\ \|x^* - x_n\|_X \leqslant \beta \|f(x_n)\|_Y . \end{cases} \tag{3}$$

In paper [2] we have studied the implications of the quality of it being an approximant sequence of the order p upon the existence of a solution x^* of the equation (1), upon establishing a convenient convergence speed towards this solution x^* of the equation (1).

In paper [3] we have linked the above issue to obtaining results that are similar to Kantorovich's theorem on Newton's method [4], [5] in which most conditions are imposed upon the initial point x_0, thus avoiding global conditions which are hard to verify. We have also taken into account I. Păvăloiu's results [6], [7], where the concept of high convergency order is introduced and we have completed them with the proof of the existence of the mapping $[f'(x_n)]^{-1}$ for any $n \in \mathbb{N}$.

We mention that by $(X,Y)^*$ we note the set of linear continuous mappings defined on X, with values in Y, and for a number $n \in \mathbb{N}$, we note with $(X^n,Y)^*$ the set of n-linear and continuous mappings defined on X^n, with values in Y. We have the following:

THEOREM 2. *Let us consider, besides the above elements, a natural number p, and $\delta > 0$; $(x_n)_{n\in\mathbb{N}} \subseteq D$.*

If:

i) *X is a Banach space and $S(x_0,\delta) \subseteq D$, $S(x_0,\delta)$ representing the ball with the centre x_0 and radius δ;*

ii) *the function $f : D \to Y$ admits Fréchet derivatives up to the p order, p included, and for $f^{(p)} : D \to (X^p,Y)^*$, $L > 0$ exists, so that the following inequality is verified for any $x, y \in D$:*

$$\left\|f^{(p)}(x) - f^{(p)}(y)\right\| \leqslant L\|x-y\|_X ;$$

iii) *$a, b \geqslant 0$ exist, so that for any $n \in \mathbb{N}$ we have:*

$$\left\|f\left(x_n + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_n)(x_{n+1}-x_n)^i\right)\right\|_Y \leqslant a\|f(x_n)\|_Y^{p+1}$$

and

$$\|f'(x_n)(x_{n+1}-x_n)\| \leqslant b\|f(x_n)\|_Y ;$$

iv) *the mapping $f'(x_0) \in (X,Y)^*$ is invertible;*

v) *if we note:*

$$\rho_0 = \|f(x_0)\|_Y, \ B_0 = \left\|\left[f'(x_0)^{-1}\right]\right\|, \ h_0 = bL_2B_0^2\rho_0, \ M = B_0 \mathrm{e}^{1+2^{-2p+3}},$$

$$\alpha = a + L\frac{(bM)^{p+1}}{(p+1)!}$$

the following inequalities are verified:

$$h_0 \leqslant \frac{1}{2}, \ \alpha^{\frac{1}{p}}\rho_0 < \frac{1}{4}, \ \delta \geqslant \frac{bM\rho_0}{1-\alpha\rho_0^p}$$

then:

j) *$x_n \in S(x_0,\delta)$, $[f'(x_n)]^{-1}$ exists and $\left\|\left[f'(x_n)^{-1}\right]\right\| \leqslant M$ for any $n \in \mathbb{N}$;*

jj) *the equation (1) admits a solution $x^* \in S(x_0,\delta)$;*

jjj) *the sequence $(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the $p+1$ order of this solution of the equation (1);*

jv) *the following evaluations take place:*

$$\max\left(\|f(x_n)\|_Y, \frac{1}{Mb}\|x_{n+1}-x_n\|_X\right) \leqslant \alpha^{\frac{(p+1)^n}{p}} \|f(x_0)\|_Y^{(p+1)^n},$$

$$\|x^* - x_n\|_X \leqslant Mb\frac{\alpha^{\frac{(p+1)^n-1}{p}} \|f(x_0)\|_Y^{(p+1)^n}}{1-\left(\alpha\|f(x_0)\|_Y^p\right)^{(p+1)^n}}.$$

For the proof, see [3].

The result above is applied in particular approximation proceedings. By taking $p = 1$ we will obtain:

COROLLARY 3. *Taking into account the data given in Theorem 1,*

i) *X is a Banach space and $S(x_0,\delta) \subseteq D$;*

ii) *the function $f : D \to Y$ admits a Fréchet derivative of the first order, and the function $f' : D \to (X,Y)^*$ verifies Lipschitz's condition, meaning that $L > 0$ exists so that for any $x, y \in D$ we have:*

$$\|f'(x) - f'(y)\| \leq L|x-y|x;$$

iii) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *verifies the equality:*
$$f'(x_n)(x_{n+1}-x_n)+f(x_n)=\theta_Y$$
for any $n\in\mathbb{N}$;

iv) *the mapping* $f'(x_0)\in(X,Y)^*$ *is invertible;*

v) *the initial point* $x_0\in D$ *verifies the inequalities:*
$$\left\|[f'(x_0)]^{-1}\right\|^2\|f(x_0)\|_Y<\frac{1}{2e^9L},$$
$$\delta\geq\frac{2e^3}{4}\left\|[f'(x_0)]^{-1}\right\|^2\|f(x_0)\|_Y;$$
then

j) $x_n\in S(x_0,\delta)$, $[f'(x_n)]^{-1}\in(Y,X)^*$ *exists,*
$$\left\|[f'(x_n)]^{-1}\right\|\leq\left\|[f'(x_0)]^{-1}\right\|e^3$$
and:
$$x_{n+1}=x_n-[f'(x_n)]^{-1}f(x_n)$$
for any $n\in\mathbb{N}$;

jj) *the equation (1) admits a solution* $x^*\in S(x_0,\delta)$;

jjj) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is an aproximant sequence of the second order of the solution* x^* *of this equation;*

jv) *the following evaluations take place:*
$$\max\left(\|f(x_n)\|_Y,\frac{1}{M}\|x_{n+1}-x_n\|_X\right)\leq\left(\frac{LM^2}{2}\right)^{2^n-1}\|f(x_0)\|_Y^{2^n};$$
$$\|x^*-x_n\|_X\leq\frac{M\rho_0\left(\frac{LM^2}{2}\rho_0\right)^{2^n}}{1-\left(\frac{LM^2}{2}\rho_0\right)^{2^n}},$$
where $M=\left\|[f'(x_0)]^{-1}\right\|e^3$ *and* $\rho_0=\|f(x_0)\|_Y$.

This corollary shows through the conclusion j) that in this case the result refers to the method of Newton-Kantorovich, the recurrence formula of which appears in this conclusion. Taking $p=2$ we can obtain the following result on Cebishev's method. We have:

COROLLARY 4. *Using the data given in the previous assertions, if:*

i) X *is a Banach space and* $S(x_0,\delta)\subseteq D$;

ii) *the function* $f:D\to Y$ *admits Fréchet derivatives up to the order 2 included, and for* $f'':D\to(X^2,Y)^*$, $L>0$ *exists so that for any* $x,y\in D$ *the following inequality is verified:*
$$\left\|f''(x)-f''(y)\right\|\leqslant L\|x-y\|_X;$$

iii) *the sequence* $(x_n)_{n\in\mathbb{N}}$, *together with an auxiliary sequence* $(y_n)_{n\in\mathbb{N}}$, *verifies the relations:*
$$f'(x_n)(x_{n+1}-x_n)+f(x_n)+\frac{1}{2}f''(x_n)y_n^2=\theta_Y,$$
$$f'(x_n)y_n+f(x_n)=\theta_Y,$$
for any $n\in\mathbb{N}$.

iv) *the mapping* $f'(x_0)\in(X,Y)^*$ *is invertible;*

v) *if we note:*
$$\rho_0=\|f(x_0)\|_Y,\ B_0=\left\|[f'(x_0)]^{-1}\right\|,\ L_2=\|f''(x_0)\|+L\delta,$$
$$M=B_0e\sqrt{e},\ b=\frac{1}{2}L_2M^2\rho_0,\ a=\frac{1}{2}L_2^2M^4(b+1),\ \alpha=a+L\frac{(bM)^3}{6};$$
the following inequalities are verified:
$$\sqrt{\alpha}\rho_0<\frac{1}{4},\ \frac{bM\rho_0}{1-\alpha\rho_0^2}\leqslant\delta\leqslant\frac{1}{L}\left(\frac{1}{2bB_0^2\rho_0}-\|f''(x_0)\|\right);$$
then:

j) $x_n\in S(x_0,\delta)$, $[f'(x_n)]^{-1}\in(X,Y)^*$ *exists,* $\left\|[f'(x_n)]^{-1}\right\|\leqslant M$ *and*
$$x_{n+1}=x_n-[f'(x_n)]^{-1}f(x_n)-\frac{1}{2}[f'(x_n)]^{-1}f''(x_n)\left\{[f'(x_n)]^{-1}f(x_n)\right\}^2$$
for any $n\in\mathbb{N}$;

jj) *the equation (1) admits a solution* $x^*\in S(x_0,\delta)$;

jjj) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is an approximant sequence of the third order of this solution;*

jv) *the following evaluations take place:*
$$\max\left(\|f(x_n)\|_Y,\frac{1}{Mb}\|x_{n+1}-x_n\|_X\right)\leqslant\alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n},$$
$$\|x^*-x_n\|_X\leqslant Mb\frac{\alpha^{\frac{3^n-1}{2}}\|f(x_0)\|_Y^{3^n}}{1-\left(\alpha\|f(x_0)\|_Y^2\right)^{3^n}},$$
for any $n\in\mathbb{N}$.

The main problem we are raising and for which we are looking for a solution in the present paper is the following: How can we influence the construction of a sequence $(x_n)_{n\in\mathbb{N}}$ using an additional sequence $(y_n)_{n\in\mathbb{N}}$, so that the order of the approximant sequence $(y_n)_{n\in\mathbb{N}}$ will grow substantially, and we will need far fewer operations in order to obtain y_n from x_n than to obtain x_{n+1}.

So, taking the data from the beginning of this paper together with the natural number p, we will consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, so that the following relations will be verified for any $n\in\mathbb{N}$:

$$\left\{\begin{array}{l} f'(x_n)(x_{n+1}-y_n)+f(y_n)=\theta_Y \\ \left\| f(x_n)+\sum\limits_{i=1}^{p}\frac{1}{i!}f^{(i)}(x_n)(y_n-x_n)^i\right\|_Y \leqslant a\|f(x_n)\|_Y^{p+1} \\ \|f'(x_n)(x_{n+1}-x_n)\|\leqslant b\|f(x_n)\|_Y \end{array}\right. \tag{4}$$

If, for any $n\in\mathbb{N}$, $[f'(x_n)]^{-1}$ exists, we can obtain from the first relation from (4):

$$x_{n+1}=y_n-[f'(x_n)]^{-1}f(y_n), \tag{5}$$

and we observe that if for any $n\in\mathbb{N}$ we choose $y_n=x_n$, we re-encounter Newton-Kantorovich's method.

It is possible for an operator $Q:X\to X$ to exist so that, for any $n\in\mathbb{N}$, $y_n=Q(x_n)$, in which case the relation (5) becomes:

$$x_{n+1}=Q(x_n)-[f'(x_n)]^{-1}f(Q(x_n)). \tag{6}$$

If $Q=I_X$, the identical operator of the space X, we re-encounter Newton-Kantorovich's method, and if for $x\in D$ we have:

$$Q(x)=x-[f'(x)]^{-1}f(x),$$

operator that we will call the operator attached to the function f, we obtain the method:

$$\begin{array}{l} x_{n+1}=x_n-[f'(x_n)]^{-1}f(x_n)- \\ -[f'(x_n)]^{-1}f\left(x_n-[f'(x_n)]^{-1}f(x_n)\right), \end{array} \tag{7}$$

method studied by J. F. Traub [7], and which will consequently bear this name.

It is clear that, if the Fréchet derivative of the second order of the mapping f for any $x\in D$ exists, we can consider:

$$Q(x)=x-[f'(x)]^{-1}f(x)-\frac{1}{2}[f'(x)]^{-1}f''(x)\left\{[f'(x)]^{-1}f(x)\right\}^2$$

obtaining the method:

$$\begin{array}{l} x_{n+1}=x_n-[f'(x_n)]^{-1}f(x_n)- \\ -\frac{1}{2}[f'(x_n)]^{-1}f''(x_n)\left\{[f'(x_n)]^{-1}f(x_n)\right\}^2-[f'(x_n)]^{-1}f(z_n), \end{array} \tag{8}$$

where

$$\begin{array}{l} z_n=x_n-[f'(x_n)]^{-1}f(x_n)- \\ -\frac{1}{2}[f'(x_n)]^{-1}f''(x_n)\left\{[f'(x_n)]^{-1}f(x_n)\right\}^2, \end{array} \tag{9}$$

which we call the Traub-Cebishev method.

In what the convergence of the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, which verify the relations (4), is concerned, we have the following:

THEOREM 5. *Let $(X,\|.\|_X)$, $(Y,\|.\|_Y)$ be two normed linear spaces, $D\subseteq X$, $f:D\to Y$, $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, verifying the relations (4), $p\in\mathbb{N}$, not null, $\delta>0$, $M>0$. If the following conditions are fulfilled:*

i) *$(X,\|.\|_X)$ is a Banach space;*

ii) *for any $x\in D$, $f^{(p)}(x)\in(X^p,Y)^*$ exists and represents the Fréchet derivative of the order p, and $f^{(p)}:D\to(X^p,Y)^*$ verifies Lipschitz's condition, meaning $L>0$ exists so that for any $x,y\in D$ we will have:*

$$\left\|f^{(p)}(x)-f^{(p)}(y)\right\|\leqslant L\|x-y\|_X;$$

iii) *$a,b\geqslant 0$ exist so that for any $n\in\mathbb{N}$ the relations (4) are verified;*

iv) *the mapping $f'(x_0)\in(X,Y)^*$ is invertible;*

v) *the inclusion $S(x_0,\delta)\subseteq D$ takes place;*

vi) *considering $L_{p+1}=L$, we introduce the sequence of constants L_0, L_1, ... L_p, L_{p+1}, so that for any $i\in\{0, 1, ..., p\}$ we will have:*

$$L_i=\left\|f^{(i)}(x_0)\right\|_Y+L_{i+1}\delta; \tag{10}$$

vii) *if we note:*

$$\rho_0=\|f(x_0)\|_Y,\ \delta_0=\|f(y_0)\|_Y,\ B_0=\left\|[f'(x_0)]^{-1}\right\|,$$

$$\begin{array}{c} \alpha=a+L\dfrac{(bM)^{p+1}}{(p+1)!},\ A=\alpha L_2M^2\left(b+\dfrac{\alpha\rho_0^p}{2}\right), \\ K=L_2\left(b+\alpha A^{-\frac{p}{p+1}}\right),\ h_i=KB_i^2\rho_i \text{ where } i\in\{1,2\}; \end{array} \tag{11}$$

the following inequalities are verified:

$$\begin{array}{c} 4A\rho_0^{p+1}<1,\ h_0\leqslant\dfrac{1}{2},\ \delta\geqslant\dfrac{M\alpha\rho_0^{p+1}}{1-\left(A\rho_0^{p+1}\right)^{p+1}}+\dfrac{2Mb\rho_0}{1-A\rho_0^{p+1}}, \\ B_0\exp\dfrac{1-\left(\dfrac{h_1}{h_0}\right)^{p+1}+\dfrac{h_1}{h_0}}{1-\left(\dfrac{h_1}{h_0}\right)^{p+1}}\leqslant M; \end{array} \tag{12}$$

then:

j) for any $n \in \mathbb{N}$, $x_n \in S(x_0,\delta)$, $[f'(x_n)]^{-1}$ exists and:

$$x_{n+1} = y_n - [f'(x_n)]^{-1} f(y_n)$$

jj) the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent and have the same limit;

jjj) if $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, then $f(x^*) = \theta_Y$;

jv) for any $n \in \mathbb{N}$, the following inequalities take place:

$$\|x_{n+1}-x_n\|_X \leqslant M\left[\frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^n} + bA^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}\right];$$

$$\|x_n-y_n\|_X \leqslant bMA^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n};$$

$$\|f(x_n)\|_Y \leqslant A^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n};$$

$$\|f(y_n)\|_Y \leqslant \frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^n};$$

(13)
$$\|x^*-x_n\|_X \leqslant M\left[\frac{\alpha}{A}\cdot\frac{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}{1-\left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}} + \frac{b}{A^{\frac{1}{p+1}}}\cdot\frac{\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}}{1-\left(A\rho_0^{p+1}\right)^{(p+2)^n}}\right];$$

$$\|x^*-y_n\|_X \leqslant M\left[\frac{\alpha}{A}\cdot\frac{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}{1-\left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}} + \frac{2b}{A^{\frac{1}{p+1}}}\cdot\frac{\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}}{1-\left(A\rho_0^{p+1}\right)^{(p+2)^n}}\right].$$

Proof. Let be the sequences: $(\rho_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$, so that:

$$\rho_0 = \|f(x_0)\|_Y,\ \delta_0 = \|f(y_0)\|_Y,\ B_0 = \left\|[f'(x_0)]^{-1}\right\|,$$

and for any $n \in \mathbb{N}$:

(14)
$$\begin{cases} h_n = KB_n^2\rho_n \\ \delta_n = \alpha\rho_n^{p+1} \\ \rho_{n+1} = A\rho_n^{p+2} \\ B_{n+1} = \dfrac{B_n}{1-h_n} \end{cases}$$

We will show that for any $n \in \mathbb{N}\cup\{0\}$, the following are true:

i) $x_n \in S(x_0,\delta)$;

ii) $[f'(x_n)]^{-1} \in (X,Y)^*$ exists, and $\left\|[f'(x_n)]^{-1}\right\| \leqslant B_n$;

iii) $\|f(x_n)\|_Y \leqslant \rho_n = A^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}$;

iv) $h_{n+1} \leqslant \min\left\{\frac{1}{2}, h_0\left(\frac{h_1}{h_0}\right)^{(p+2)^n}\right\}$;

v) $B_0 \leqslant B_n \leqslant M$;

vi) $y_n \in S(x_0,\delta)$;

vii) $\|f(y_n)\|_Y \leqslant \delta_n = \frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^n}$.

We will use the method of mathematical induction.

From the hypotheses of the theorem, with the notations we have introduced, we deduce that i), ii), iii), v) are evidently true for $n = 0$.

In what iv) is concerned, we notice that:

$$h_1 = KB_1^2\rho_1 = K\frac{B_0^2}{(1-h_0)^2}A\rho_0^{p+2} = A\frac{h_0}{(1-h_0)^2}\rho_0^{p+1}.$$

From $h_0 \leqslant \frac{1}{2}$ we deduce that $\frac{h_0}{(1-h_0)^2} \leqslant 2$, so $h_1 \leqslant 2A\rho_0^{p+1}$. As $\rho_0 \leqslant (4A)^{-\frac{1}{p+1}}$ we deduce that $A\rho_0^{p+1} \leqslant \frac{1}{4}$ and thus $h_1 \leqslant \frac{1}{2}$. Evidently $h_1 = h_0\left(\frac{h_1}{h_0}\right) = h_0\left(\frac{h_1}{h_0}\right)^{(p+2)^0}$, so iv) is true for $n = 0$.

For vi), taking into account the existence of the mapping $[f'(x_0)]^{-1}$, we have

$$\|y_0 - x_0\|_X = \left\|[f'(x_0)]^{-1} f'(x_0)(y_0-x_0)\right\|_X \leqslant$$
$$\leqslant \left\|[f'(x_0)]^{-1}\right\| \left\|f'(x_0)(y_0-x_0)\right\|_Y \leqslant B_0 b\|f(x_0)\|_Y = B_0 b\rho_0 \leqslant$$
$$\leqslant Mb\rho_0 \leqslant Mb\frac{\rho_0}{1-A\rho_0^{p+1}} \leqslant \delta,$$

so $y_0 \in S(x_0,\delta)$.

Finally, for vii) we will have:

$$\|f(y_0)\|_Y = \delta_0 = \alpha\rho_0^{p+1} = \frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^0}.$$

So i)-vii) are all true for $n = 0$.

Let us now suppose that they are true for any $n \leqslant k$, aiming to demonstrate them for $n = k+1$. So:

i) for any $n \leqslant k$ we have:

$$\|x_{n+1}-x_n\|_X \leqslant \|x_{n+1}-y_n\|_X + \|y_n - x_n\|_X$$

and

$$\|x_{n+1}-y_n\|_X \leqslant \left\|[f'(x_n)]^{-1}\right\| \cdot \|f'(x_n)(x_{n+1}-y_n)\|_Y \leqslant$$

$$\leqslant B_n\|f(y_n)\|_Y \leqslant B_n\delta_n \leqslant M\frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^n},$$

but

$$\|y_n-x_n\|_X \leqslant \left\|[f'(x_n)]^{-1}\right\| \cdot \|f'(x_n)(y_n-x_n)\|_Y \leqslant$$

$$\leqslant bB_n\rho_n \leqslant bMA^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}.$$

So:

$$\text{(15)}\qquad \|x_{n+1}-x_n\|_X \leqslant M\left[\frac{\alpha}{A}\left(A\rho_0^{p+1}\right)^{(p+2)^n} + bA^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}\right].$$

From here:

$$\|x_{k+1}-x_0\|_X \leqslant \sum_{n=0}^{k}\|x_{n+1}-x_n\|_X \leqslant$$

$$\leqslant M\frac{\alpha}{A}\sum_{n=0}^{k}\left(A\rho_0^{p+1}\right)^{(p+2)^n} + MbA^{-\frac{1}{p+1}}\sum_{n=0}^{k}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}.$$

As $A\rho_0^{p+1} < 1$ and so $A^{\frac{1}{p+1}}\rho_0 < 1$ we will have:

$$\sum_{n=0}^{k}\left(A\rho_0^{p+1}\right)^{(p+2)^n} = A\rho_0^{p+1}\sum_{n=0}^{k}\left(A\rho_0^{p+1}\right)^{(p+2)^n-1} <$$

$$< A\rho_0^{p+1}\sum_{n=0}^{k}\left[\left(A\rho_0^{p+1}\right)^{p+1}\right]^n < \frac{A\rho_0^{p+1}}{1-\left(A\rho_0^{p+1}\right)^{p+1}}$$

and likewise:

$$\sum_{n=0}^{k}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n} < \frac{A^{\frac{1}{p+1}}\rho_0^{p+1}}{1-A\rho_0^{p+1}}.$$

So:

$$\|x_{k+1}-x_0\|_X < \frac{M\ \alpha\ \rho_0^{p+1}}{1-\left(A\rho_0^{p+1}\right)^{p+1}} + \frac{M\ b\ \rho_0}{1-A\rho_0^{p+1}} \leq \delta,$$

so $x_{k+1} \in S(x_0,\delta)$.

ii) Let $H_k = [f'(x_k)]^{-1}(f'(x_k)-f'(x_{k+1})) = I_X - [f'(x_k)]^{-1}f'(x_{k+1})$ so

$$f'(x_{k+1}) = f'(x_k)(I_X - H_k)$$

and thus:

$$\|H_k\| \leqslant \left\|[f'(x_k)]^{-1}\right\| \cdot \|f'(x_k)-f'(x_{k+1})\| \leqslant$$

$$\leqslant B_kL_2\|x_{k+1}-x_k\|_X \leqslant B_kL_2\|x_{k+1}-y_k\|_X + \|y_k-x_k\|_X \leqslant$$

$$\leqslant B_kL_2\left(b_k\|f(y_k)\|_Y + B_kb\|f(x_k)\|_Y\right) = L_2\left(b\delta_k+\alpha\delta_k\right)B_k^2 =$$

$$= L_2\left(b+\alpha\delta_k^p\right)B_k^2\rho_k.$$

As $\rho_k = A^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^k}$ and $A^{\frac{1}{p+1}}\rho_0 < 1$ we deduce that $\rho_k^p < A^{-\frac{p}{p+1}}$, so:

$$\|H_k\| < L_2\left(b+\alpha A^{-\frac{p}{p+1}}\right)B_h^2\rho_k = KB_h^2\rho_k = h_k \leqslant \frac{1}{2},$$

from where we deduce that $(I_X-H_k)^{-1} \in (X,X)^*$ exists and:

$$\left\|(I_X-H_k)^{-1}\right\| \leqslant \frac{1}{1-\|H_k\|} < \frac{1}{1-h_k}.$$

From the existence of the mapping $[f'(x_k)]^{-1}$ we deduce the existence of the mapping $[f'(x_{k+1})]^{-1}$ and:

$$[f'(x_{k+1})]^{-1} = (I_X-H_k)^{-1}[f'(x_k)]^{-1}$$

with:

$$\left\|[f'(x_{k+1})]^{-1}\right\| \leqslant \left\|(I_X-H_k)^{-1}\right\| \cdot \left\|[f'(x_k)]^{-1}\right\| \leqslant \frac{B_k}{1-h_k} = B_{k+1}.$$

So ii) is true.

iii) We have:

$$\|f(x_{k+1})\|_Y = \|f(x_{k+1}) - f(y_k) - f'(x_k)(x_{k+1}-y_k)\|_Y \leqslant$$

$$\leqslant \|f(x_{k+1}) - f(y_k) - f'(y_k)(x_{k+1}-y_k)\|_Y +$$

$$+ \|(f'(y_k)-f'(x_k))(x_{k+1}-y_k)\|_Y \leqslant$$

$$\leqslant \frac{L_2}{2}\|x_{k+1}-y_k\|_X^2 + L_2\|y_k-x_k\|_X \cdot \|x_{k+1}-y_k\|_X \leqslant$$

$$\leqslant L_2\left(\frac{1}{2}B_k^2\delta_k^2 + bB_k^2\rho_k\delta_k\right) \leqslant \alpha L_2\left(\frac{\alpha}{2}\rho_k^p + b\right)B_k^2\rho_k^{p+2}.$$

As $A^{\frac{1}{p+1}}\rho_0 < 1$, we deduce that $\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n} \leqslant A^{\frac{1}{p+1}}\rho_0$, and as $B_k \leqslant M$ we deduce that:

$$\|[f(x_{k+1})]\|_Y \leqslant \alpha L_2 \left(b + \frac{\alpha\rho_0^p}{2}\right) M^2 \rho_k^{p+2} = \rho_{k+1}.$$

From here:

$$A^{\frac{1}{p+1}}\rho_{k+1} = \left(A^{\frac{1}{p+1}}\rho_k\right)^{p+2}$$

and from the hypothesis of the induction we deduce that:

$$\|f(x_{k+1})\|_Y \leqslant \rho_{k+1} \leqslant A^{-\frac{1}{p+1}} \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^{k+1}}.$$

iv) We have:

$$\begin{aligned} h_{k+1} &= KB_{k+1}^2\rho_{k+1} = K\frac{B_k^2}{(1-h_k)}A\rho_k^{p+2} = \\ &= A\frac{h_k}{(1-h_k)^2}\rho_k^{p+1} \leqslant 2A\rho_k^{p+1} \leqslant 2\left(A\rho_0^{p+1}\right)^{(p+2)^k} \end{aligned}$$

and as $\rho_0 \leqslant (4A)^{-\frac{1}{p+1}}$ we have $A\rho_0^{p+1} \leqslant \frac{1}{4}$, so $h_{k+1} \leqslant 2\left(\frac{1}{4}\right)^{(p+2)^k}$, so $h_{k+1} \leqslant \frac{1}{2}$.

Also:

$$h_{k+1} = A\frac{h_k}{(1-h_k)^2}\rho_k^{p+1} = A\frac{h_k}{(1-h_k)^2}\cdot\frac{h_k^{p+1}}{K^{p+1}B_0^{2(p+1)}},$$

and, as $B_k \geqslant B_0$, we deduce that:

$$h_{k+1} \leqslant \frac{A}{(KB_0^2)^{p+1}}\cdot\frac{1}{(1-h_k)^2}h_k^{p+2} \leqslant \frac{4A}{(KB_0^2)^{p+1}}h_k^{p+2}.$$

From the hypotheses we have $4A \leqslant \frac{1}{\rho_0^{p+1}}$, so $h_{k+1} \leqslant \frac{1}{(KB_0^2\rho_0)^{p+1}}\cdot h_k^{p+2}$, so $\frac{h_{k+1}}{h_0} \leqslant \left(\frac{h_k}{h_0}\right)^{p+2}$ and from the hypothesis of the induction we have:

$$h_{k+1} \leqslant h_0\left(\frac{h_1}{h_0}\right)^{(p+2)^k}$$

so:

$$h_{k+1} \leqslant \min\left\{\frac{1}{2}, h_0\left(\frac{h_1}{h_0}\right)^{(p+2)^k}\right\}.$$

v) From $B_{k+1} = \frac{B_k}{1-h_k}$ we deduce $\frac{B_{k+1}}{B_k} = \frac{1}{1-h_k}$.

As $0 < h_k \leqslant \frac{1}{2}$, we have the fact that $\frac{1}{1-h_k} > 1$, $\frac{B_{k+1}}{B_k} \geqslant 1$ and taking into account the hypothesis of the induction $B_k \geqslant B_0$, we have $B_{k+1} \geqslant B_0$.

Evidently:

$$B_{k+1} = \frac{B_0}{(1-h_0)(1-h_1)\dots(1-h_k)}$$

and

$$\frac{1}{(1-h_0)(1-h_1)\dots(1-h_k)} \leqslant \left[1 + \frac{1}{k+1}\sum_{i=0}^{k}\frac{h_i}{1-h_i}\right]^{k+1}.$$

As we have seen, $\frac{h_{i+1}}{h_0} \leqslant \left(\frac{h_i}{h_0}\right)^{p+2}$ for any $i \leqslant k$. From here we deduce that for $n = 0$, $h_1 \leqslant h_0$. From the same relation, for $i = 1$ we have: $\frac{h_2}{h_0} \leqslant \left(\frac{h_1}{h_0}\right)^{p+2} \leqslant 1$ and $h_2 \leqslant h_0$. Through the induction we will thus deduce that $h_i \leqslant h_0$ for any $i \leqslant k+1$, so $\frac{1}{1-h_i} \leqslant \frac{1}{1-h_0}$, so:

$$\frac{1}{(1-h_0)(1-h_1)\dots(1-h_k)} \leqslant \left[1 + \frac{1}{(k+1)(1-h_0)}\sum_{i=0}^{k}h_i\right]^{k+1}.$$

Also:

$$\begin{aligned} \sum_{i=0}^{k}h_i &= h_0 + \sum_{i=0}^{k}h_i \leqslant h_0 + h_0\sum_{i=1}^{k}\left(\frac{h_1}{h_0}\right)^{(p+2)^{i-1}} < \\ &< h_0 + h_0\sum_{i=1}^{k}\left[\left(\frac{h_1}{h_0}\right)^{p+1}\right]^{i-1} < h_0\left[1 + \frac{\frac{h_1}{h_0}}{1-\left(\frac{h_1}{h_0}\right)^{p+1}}\right], \end{aligned}$$

and so:

$$\frac{1}{1-h_0}\sum_{i=0}^{k}h_i \leqslant \frac{h_0}{1-h_0}\left[1 + \frac{\frac{h_1}{h_0}}{1-\left(\frac{h_1}{h_0}\right)^{p+1}}\right] \leqslant \frac{1-\left(\frac{h_1}{h_0}\right)^{p+1} + \frac{h_1}{h_0}}{1-\left(\frac{h_1}{h_0}\right)^{p+1}}.$$

That condition $\dfrac{h_1}{h_0} < 1$ that rises gives:

$$\frac{h_1}{h_0} = \frac{KB_1^2\rho_1}{KB_0^2\rho_0} \leqslant \left(\frac{1}{1-h_0}\right)^2 A\rho_0^{p+1} \leqslant 4A\rho_0^{p+1} < 1.$$

So we have:

$$B_{k+1} \leqslant B_0 \left[1 + \frac{1}{k+1} \frac{1 - \left(\frac{h_1}{h_0}\right)^{p+1} + \frac{h_1}{h_0}}{1 - \left(\frac{h_1}{h_0}\right)^{p+1}}\right]^{p+1} \leqslant$$

$$\leqslant B_0 \exp \frac{1 - \left(\frac{h_1}{h_0}\right)^{p+1} + \frac{h_1}{h_0}}{1 - \left(\frac{h_1}{h_0}\right)^{p+1}} = M,$$

So we have: $B_0 \leqslant B_{k+1} \leqslant M$.
vi) Just like for $n = k$ we have:

$$\|y_{k+1} - x_{k+1}\|_X \leqslant B_{k+1} b\rho_{k+1} \leqslant bMA^{-\frac{1}{p+1}} \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^{k+1}} <$$

$$< bMA^{-\frac{1}{p+1}} \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)} < bM\rho_0 \leqslant \frac{bM\rho_0}{1 - A\rho_0^{p+1}},$$

so:

$$\|y_{k+1} - x_0\|_X \leqslant \|y_{k+1} - x_{k+1}\|_X + \|x_{k+1} - x_0\|_X < \frac{M\alpha\rho_0^{p+1}}{1 - \left(A\rho_0^{p+1}\right)^{p+1}} +$$

$$+ \frac{2Mb\rho_0}{1 - A\rho_0^{p+1}} = \delta,$$

thus: $y_{k+1} \in S(x_0, \delta)$.

vii) As we know that $x_{k+1},\ y_{k+1} \in S(x_0, \delta)$ we can deduce the inequality:

$$\left\| f(y_{k+1}) - f(x_{k+1}) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_{k+1})(y_{k+1} - x_{k+1})^i \right\|_Y \leq$$

$$\leq \frac{L}{(p+1)!} \|y_{k+1} - x_{k+1}\|_Y^{p+1} \leq L \frac{(bB_{k+1})^{p+1}}{(p+1)!} \|f(x_{k+1})\|_Y^{p+1} \leq$$

$$\leq L \frac{(bM)^{p+1}}{(p+1)!} \|f(x_{k+1})\|_Y^{p+1},$$

so

$$\|f(y_{k+1})\|_Y \leq \left\| f(y_{k+1}) - f(x_{k+1}) - \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_{k+1})(y_{k+1} - x_{k+1})^i \right\|_Y +$$

$$+ \left\| f(x_{k+1}) + \sum_{i=1}^{p} \frac{1}{i!} f^{(i)}(x_{k+1})(y_{k+1} - x_{k+1})^i \right\|_Y \leq$$

$$\leq \left[a + L\frac{(bM)^{p+1}}{(p+1)!}\right] \|f(x_{k+1})\|_Y^{p+1} \leq \rho_{k+1}^{p+1}$$

and so $\|f(y_{k+1})\|_Y \leq \delta_{k+1}$.

From the previous relations, using the principle of mathematical induction i)–vii) proved to be true for any $n \in \mathbb{N}$.

From here we will show that $(x_n)_{n\in\mathbb{N}}$ is fundamental. Indeed:

$$\|x_{n+m} - x_n\|_X \leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\|_X \leq$$

$$\leq M \left[\frac{\alpha}{A} \sum_{i=n}^{n+m-1} \left(A\rho_0^{p+1}\right)^{(p+2)^i} + \frac{b}{A^{\frac{1}{p+1}}} \sum_{i=n}^{n+m-1} \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^i}\right].$$

Evidently:

$$\sum_{i=n}^{n+m-1} \left(A\rho_0^{p+1}\right)^{(p+2)^i} = \left(A\rho_0^{p+1}\right)^{(p+2)^n} \sum_{i=0}^{m-1} \left(A\rho_0^{p+1}\right)^{(p+2)^{n+j} - (p+2)^n} <$$

$$< \left(A\rho_0^{p+1}\right)^{(p+2)^n} \sum_{i=0}^{m-1} \left[\left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}\right]^j < \frac{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}{1 - \left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}}$$

and likewise:

$$\sum_{i=n}^{n+m-1} \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^i} < \frac{\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}}{1 - \left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}}.$$

So:

$$\|x_{n+m} - x_n\|_X < M \left[\frac{\alpha}{A} \cdot \frac{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}{\left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}} + \right.$$

(16)

$$\left. + \frac{b}{A^{\frac{1}{p+1}}} \cdot \frac{\left(A\rho_0^{p+1}\right)^{\frac{(p+2)^n}{p+1}}}{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}\right].$$

As $A\rho_0^{p+1} < 1$, we deduce that the limit of the right side of the previous inequality in 0, showing the fact that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space X, so this sequence is convergent and $x^* \in X$ exists so that $x^* = \lim_{n\to\infty} x_n$.

If in the relation (16) we now make $m \to \infty$ and then we consider the fact that $\left(A\rho_0^{p+1}\right)^{\frac{(p+2)^n}{p+1}} < \left(A\rho_0^{p+1}\right)^{(p+2)^{n-1}}$, we will obtain the fifth of the relations (13) of the conclusion.

Now we put $n = 0$ into this inequality, thus obtaining:

$$\|x^* - x\|_X \le \frac{M\alpha\rho_0^{p+1}}{\left(A\rho_0^{p+1}\right)^{p+1}} + \frac{Mb\rho_0}{1 - A\rho_0^{p+1}} < \delta,$$

and so $x^* \in S(x_0, \delta)$.

As

$$0 \le \|f(x_n)\|_Y \le \rho_n = A^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n},$$

and the limit of the expression on the right is 0, so $\lim_{n\to\infty}\|f(x_n)\|_Y = 0$, from where $f(x^*) = \theta_Y$.

From:

$$\|y_n - x_n\|_X \le b\,M\,A^{-\frac{1}{p+1}}\left(A^{\frac{1}{p+1}}\rho_0\right)^{(p+2)^n}$$

we evidently deduce that $\lim_{n\to\infty}(y_n - x_n) = \theta_X$, so $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = x^*$.

If we take:

$$\|x_{n+m} - y_n\|_X \le \|x_{n+m} - x_n\|_X + \|x_n - y_n\|_X \le \|x_{n+m} - x_n\|_X <$$
$$< M\left[\frac{\alpha}{A}\frac{\left(A\rho_0^{p+1}\right)^{(p+2)^n}}{\left(A\rho_0^{p+1}\right)^{(p+1)(p+2)^n}} + \frac{2b}{A^{\frac{1}{p+1}}}\frac{\left(A\rho_0^{p+1}\right)^{\frac{(p+2)^n}{p+1}}}{1-\left(A\rho_0^{p+1}\right)^{(p+2)^n}}\right],$$

we now make $m \to \infty$ here, and we will obtain the last of the inequalities (13) of the conclusion.

So the theorem has been proven.

We will illustrate the importance of the previous theorem through applications for the case of particular iterative proceedings.

Let us consider the pair of sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ that verify the relations:

$$\text{(17)}\qquad \begin{cases} f'(x_n)(y_n - x_n) + f(x_n) = \theta_Y, \\ f'(x_n)(x_{n+1} - y_n) + f(y_n) = \theta_Y. \end{cases}$$

Clearly, if $[f'(x_n)]^{-1}$ exists for any $n \in \mathbb{N}$, this condition being fulfilled if the conditions of the theorem 5 are verified, then the relations (17) will give:

$$\text{(18)}\qquad \begin{cases} y_n = x_n - [f'(x_n)]\,f(x_n), \\ x_{n+1} = x_n - [f'(x_n)]\,f(x_n) - \\ \quad - [f'(x_n)]^{-1} f(x_n - [f'(x_n)]\,f(x_n)), \end{cases}$$

so in this case the iterative method (17) is reduced to the iterative method (7).

For the method generated by the relations (17) we will show that the hypotheses of the theorem 5 are verified for $p = 1$. So, if we suppose that the Fréchet derivative of the first order exists in any point of the set D that the constant $L > 0$ exists and has the property that for any $x, y \in D$ we have:

$$\left\|f'(x) - f'(y)\right\| \le L\,\|x - y\|$$

and that the mapping $[f'(x_0)]^{-1}$ is invertible, we notice that the relations (4) are verified for $a = 0$ and $b = 1$.

If we calculate the constants for this particular case we will have:

$$\text{(19)}\qquad \begin{aligned} &\|f(x_n)\|_Y \le \frac{\left(\sqrt{A\rho_0}\right)^{3^n}}{A}, \\ &\|x_{n+1} - x_n\|\,x \le M\left[\frac{\alpha}{A}\left(A\rho_0^2\right)^{3^n} + bA^{-\frac{1}{2}}\left(A\rho_0\right)^{3^n}\right], \end{aligned}$$

relations which show that the order of convergence of this method is 3, the same as the one of the method the convergence of which is given by Corollary 4, but presenting the advantage that in this case the calculations are much simpler.

We can still improve the convergence speed through the method by which the trio of sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subseteq X$ verifies the relations:

$$\text{(20)}\qquad \begin{cases} f'(x_n)\,z_n + f(x_n) = \theta_Y, \\ f'(x_n)(y_n - x_n) + f(x_n) + \frac{1}{2}f''(x_n)\,z_n^2 = \theta_Y, \\ f'(x_n)(x_{n+1} - y_n) + f(y_n) = \theta_Y. \end{cases}$$

If the hypotheses of the theorem 5 are fulfilled $[f'(x_n)]^{-1}$ exists for any $n \in \mathbb{N}$ and thus the recurrence relations from (20) become:

$$\text{(21)}\qquad \begin{cases} z_n = -[f'(x_n)]^{-1} f(x_n) \\ y_n = x_n - [f'(x_n)]^{-1} f(x_n) - \frac{1}{2}[f'(x_n)]^{-1} f''(x_n)\,z_n^2 \\ x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n) - \frac{1}{2}[f'(x_n)]^{-1} f''(x_n)\,z_n^2 - \\ \quad - [f'(x_n)]^{-1} f(y_n) \end{cases}$$

So we are facing the case of the method (8).

We will show that $p = 2$ is the number for which the hypotheses of the theorem 5 are verified in the case of the method (21).

We suppose that the Fréchet derivative up to the second order, including it, exists on any point $x \in D$ and that the constant $L > 0$ for which:

$$\left\| f''(x) - f''(y) \right\| \leq L \left\| x - y \right\|_X$$

is valid for any $x, y \in D$ also exists.

We keep in mind the fact that, unlike the method in Corollary 4, in the method (21) the role of y_n is taken by z_n; so, for a given x_n, the role of x_{n+1} is taken by y_n. According to the results from the proof of the theorem 4 in paper [3], we conclude that the following are true for any $n \in \mathbb{N}$.

a) $x_n \in S(x_0, \delta)$

b) $[f'(x_n)]^{-1} \in (Y, X)^*$ exists and $\left\| [f'(x_n)]^{-1} \right\| \leq B_n$;

c) $\|f(x_n)\|_Y \leq \rho_n \leq \frac{1}{A^{\frac{1}{3}}}\left(A^{\frac{1}{3}}\rho_0\right)^{4^n}$;

d) $h_n \leq \min\left\{\frac{1}{2}, \frac{1}{\beta^{\frac{1}{3}}}(\beta h_0)^{4^n}\right\}$, where $\beta = \frac{1}{16 h_0^2}$;

e) $B_0 \leq B_n \leq M$;

f) $\|f'(x_n)(y_n - x_n)\|_Y \leq b \|f(x_n)\|_Y$;

g) $\left\| f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2} f''(x_n)(y_n - x_n)^2 \right\|_Y \leq a \|f(x_n)\|_Y^3$.

Here the sequences $(\rho_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$, $(h_n)_{n \in \mathbb{N}}$ have the significations from the theorem 5 with $p = 3$, so if we consider $M_0 = \left\| [f'(x_n)]^{-1} \right\| e\sqrt{e}$ and we proceed just as in the demonstration of the theorem 4, from (3) we will deduce:

$$b = 1 + \frac{1}{2} L_2 M_0^2 \rho_0, \quad a = \frac{1}{2} L_2^2 M_0^4 (b + 1).$$

The relations *a)*–*g)* are true if for $\alpha = a + L\frac{(bM)^3}{6}$, $A = \alpha L_2 M^2 \left(b + \frac{\alpha \rho_0^2}{2}\right)$, $K = L_2\left(b + \frac{\alpha}{A^{\frac{2}{3}}}\right)$ and $h_i = K B_i^2 \rho_i$ with $i \in \{1, 2\}$ the following inequalities are verified:

$$4A\rho_0^3 < 1, \quad h \leq \frac{1}{2}, \quad \delta \geq \frac{M\,\alpha\,\rho_0^3}{1 - (A\rho_0^3)^3} + \frac{2\,M\,b\,\rho_0}{1 - A\rho_0^3},$$

$$B_0 \exp \frac{1 - \left(\frac{h_1}{h_0}\right)^3 + \frac{h_1}{h_0}}{1 - \left(\frac{h_1}{h_0}\right)^3} \leq M.$$

So we notice that the hypotheses of the theorem 5 are verified for $p = 2$ and thus, according to the same theorem, it results that, for any $n \in \mathbb{N}$, $[f'(x_n)]^{-1}$ exists and thus the sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ which verify

the relations (20) are given by the recurrence relations (21), they converge towards a solution x^* of the equation (1) and for any $n \in \mathbb{N}$ the following inequalities are verified:

$$(22) \quad \begin{cases} \|x_{n+1} - x_n\|_X \leq \frac{M\left(\alpha + bA^{\frac{2}{3}}\right)}{A}\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}, \\ \|x_n - y_n\|_X \leq \frac{bM}{A^{\frac{1}{3}}}\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}, \\ \|f(x_n)\|_Y \leq \frac{\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}}{A^{\frac{1}{3}}}, \\ \|f(y_n)\|_Y \leq \frac{\alpha\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}}{A^{\frac{4}{3}}}, \\ \|x^* - x_n\|_X \leq \frac{M\left(\alpha + bA^{\frac{2}{3}}\right)}{A} \cdot \frac{\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}}{1 - \left(A\rho_0^3\right)^{4^n}}, \\ \|x^* - y_n\|_X \leq \frac{M\left(\alpha + bA^{\frac{2}{3}}\right)}{A} \cdot \frac{\left(\rho_0 A^{\frac{1}{3}}\right)^{4^n}}{1 - \left(A\rho_0^3\right)^{4^n}}. \end{cases}$$

These relations show that the order of the method we have studied is 4, so we have obtained an acceleration of the convergency speed.

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