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THE CONCAVITY OF SOME SPECIAL FUNCTIONS [1]

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Abstract. In this paper, one studies the concavity of the functions $g : D \to \mathbb{R}$ of the form

$$g(x) = \sum_{i=1}^{p} a_i f_i(x) + \left(\sum_{i=1}^{p} f_i(x) \right) \ln \sum_{i=1}^{p} f_i(x) - \sum_{i=1}^{p} f_i(x) \ln f_i(x), \ x \in D,$$

where $D \subseteq \mathbb{R}^n$ is a nonempty convex set, $f_1, \ldots, f_p : D \to]0, \infty[$ are concave functions and $a_1, \ldots, a_p$ are real numbers.

As known, the necessary conditions (without any differentiability hypothesis on the functions) and the sufficient conditions (with differentiability hypothesis on the functions) for the optimal solutions of the optimization problems are formulated under concavity (or generalized concavity) assumptions on the objective function and restrictions.

In this paper, one studies the concavity of a function by showing that its values are the optimum values of a convex optimization problem. Using this idea H. P. Benson and G. M. Boger, [3] show that, if $D \subseteq \mathbb{R}^n$ is a nonempty convex set and $f_1, \ldots, f_p : D \to]0, +\infty[$ are concave functions, then the function $\alpha : D \to \mathbb{R}$ defined for each $x \in D$, by

$$\alpha(x) = \left(\prod_{i=1}^{p} f_i(x) \right)^{1/p}$$

is a concave function. From this it follows that, if $D \subseteq \mathbb{R}^n$ is a nonempty convex set and $f_1, \ldots, f_p : D \to]0, +\infty[$ are concave functions, then the function $\beta : D \to \mathbb{R}$, defined for each $x \in D$ by

$$\beta(x) = \prod_{i=1}^{p} f_i(x)$$

is a quasiconcave function.

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Using the same idea, D. I. Duca [4] shows that, if $D \subseteq \mathbb{R}^n$ is a nonempty convex set, and $f_1, \dots, f_p : D \to]0, +\infty[$ are concave functions, then, for each $r > 1$ and $a_1, \dots, a_p > 0$, the function $\gamma : D \to \mathbb{R}$ defined, for each $x \in D$, by

$$\gamma(x) = \left(\sum_{i=1}^{p} a_i (f_i(x))^{1/r} \right)^r$$

is a concave function.

In this paper, one studies the concavity of the functions $g : D \to \mathbb{R}$ defined, for each $x \in D$, by

$$g(x) = \sum_{i=1}^{p} a_i f_i(x) + \left(\sum_{i=1}^{p} f_i(x) \right) \ln \sum_{i=1}^{p} f_i(x) - \sum_{i=1}^{p} f_i(x) \ln f_i(x),$$

where $D \subseteq \mathbb{R}^n$ is a nonempty convex set, $f_1, \dots, f_p : D \to]0, +\infty[$ are concave functions and $a_1, \dots, a_p > 0$ are real numbers.

We need the following lemma.

LEMMA 1. *Let $a = (a_1, \dots, a_p) > 0$ and $b = (b_1, \dots, b_p) > 0$ which satisfy the following inequalities*

$$\left(\sum_{i=1}^{p} a_i \right) b_k \geq a_k, \; k \in \{1, \dots, p\}. \tag{1}$$

Then the optimization problem

$$\min (a_1 z_1 + \dots + a_p z_p), \tag{P}$$

subject to

$$b_1 e^{-z_1} + \dots + b_p e^{-z_p} \leq 1,$$
$$z_i > 0, \; i \in \{1, \dots, p\}$$

has a unique optimal solution $z^0 = (z_1^0, \dots, z_p^0) \in \mathbb{R}^p$, given by

$$z_i^0 = \ln \left(\sum_{i=1}^{p} a_i \right) + \ln b_i - \ln a_i, \; i \in \{1, \dots, p\}. \tag{2}$$

Proof. Let $\varphi : \mathbb{R}^p \to \mathbb{R}$ be defined, for each $z = (z_1, \dots, z_p) \in \mathbb{R}^p$, by

$$\varphi(z) = b_1 e^{-z_1} + \dots + b_p e^{-z_p} - 1.$$

Obviously, the function φ is convex and differentiable on $\mathbb{R}^p$.

Let us suppose that the problem (P) has an optimal solution $z^0 = (z_1^0, \dots, z_p^0) > 0$. Since the function φ satisfies Slater's constraint qualification, in view of Karush-Kuhn-Tucker necessary optimality theorem [5, pp. 109–110], there exists a nonnegative number v, such that

$$a_i - b_i e^{-z_i^0} v = 0, \; i \in \{1, \dots, p\}, \tag{3}$$

$$v \left(\sum_{i=1}^{p} b_i e^{-z_i^0} - 1 \right) = 0. \tag{4}$$

If $v = 0$, then, from (3), it follows that $a_i = 0$, $i \in \{1, ..., p\}$, which contradicts $a > 0$. Hence $v > 0$. Then, from (4), we deduce that

$$\sum_{i=1}^{p} b_i e^{-z_i^0} = 1. \tag{5}$$

Now, from (3) it follows that

$$z_i^0 = \ln \frac{v b_i}{a_i}, \; i \in \{1, ..., p\}. \tag{6}$$

By substitution in (5), it implies that

$$v = \sum_{i=1}^{p} a_i. \tag{7}$$

Now, from (6) and (7), it follows (2). Therefore, if the problem (P) has an optimal solution $z^0 = (z_1^0, ..., z_p^0) \in \mathbb{R}^p$, then this is unique and given by (2).

On the other hand, the problem (P) is convex and, for $z^0 = (z_1^0, ..., z_p^0) \in \mathbb{R}^p$ given by (2), there exists a nonnegative number v given by (7) such that the Karush-Kuhn-Tucker conditions (3)–(4) are hold. Then, in view of the Karush-Kuhn-Tucker sufficient optimality theorem [5, pp. 93–95], the point z^0 is an optimal solution of the problem (P).

Remark 1. If $b_k \geqslant 1$, for all $k \in \{1, ..., p\}$ then the condition (1) is satisfied.

Using Lemma 1, we can state the following theorem.

THEOREM 1. *Let $D \subseteq \mathbb{R}^n$ be a nonempty convex set, let $f_1, ..., f_p : D \to]0, \infty[$ be concave functions and $a_1, ..., a_p$ be real numbers such that*

$$\left(\sum_{i=1}^{p} f_i(x) \right) e^{a_k} \geqslant f_k(x), \text{ for any } k \in \{1, ..., p\} \text{ and } x \in D. \tag{8}$$

Let $g : D \to \mathbb{R}$ be the function defined for each $x \in D$, by

$$g(x) = \sum_{i=1}^{p} a_i f_i(x) + \left(\sum_{i=1}^{p} f_i(x) \right) \ln \sum_{i=1}^{p} f_i(x) - \sum_{i=1}^{p} f_i(x) \ln f_i(x).$$

Then g is a concave function.

Proof. Let $b_i = e^{a_i}$, $i \in \{1, ..., p\}$. Then $b_1, ..., b_p > 0$. Let us consider the function $h : D \to \mathbb{R}$ defined, for each $x \in D$, by

$$h(x) = \min\left\{\sum_{i=1}^{p} z_i f_i(x) : (z_1, ..., z_p) \in Z\right\}, \tag{9}$$

where

$$Z = \left\{(z_1, ..., z_p) \in \operatorname{int}\mathbb{R}_+^p : \sum_{i=1}^{p} b_i e_i^{-z_i} \leq 1\right\}.$$

From Lemma 1, since $f_1, ..., f_p$ are strictly positive on D, and

$$\left(\sum_{i=1}^{p} f_i(x)\right) e^{a_k} \geq f_k(x), \text{ for any } k \in \{1, ..., p\} \text{ and } x \in D,$$

it follows that the minimum in (9) exists and is finite for each $x \in D$. If, for each $z = (z_1, ..., z_p) \in Z$, we define the function $h_z : D \to \mathbb{R}$,

$$h_z(x) = \sum_{i=1}^{p} z_i f_i(x), \ x \in D$$

then, for each $x \in D$, $h(x)$ may also be written as

$$h(x) = \min\{h_z(x) : (z_1, ..., z_p) \in Z\}. \tag{10}$$

Obviously, for each $(z_1, ..., z_p) \in Z$, the function h_z is concave. From this and (10) we deduce that the function h is also concave.

To complete the proof, we will show that, for each $x \in D$, we have $h(x) = g(x)$. For this aim, fix $x \in D$ and let $z(x) = (z_1(x), ..., z_p(x)) \in Z$ be an optimal solution of the problem (9). From the Karush-Kuhn-Tucker theorem, it follows that there exists a nonnegative number $v(x)$, such that

$$f_i(x) - b_i v(x) e^{-z_i(x)} = 0, \ i \in \{1, ..., p\}, \tag{11}$$

$$\left(\sum_{i=1}^{p} b_i e^{-z_i(x)} - 1\right) v(x) = 0. \tag{12}$$

If $v(x) = 0$, then, from (11), it follows that $f_i(x) = 0$, for each $i \in \{1, ..., p\}$, which contradicts $f_i(x) > 0$, for all $x \in D$ and $i \in \{1, ..., p\}$. Hence $v(x) > 0$. Then, from (12), we deduce that

$$\sum_{i=1}^{p} b_i e^{-z_i(x)} = 1. \tag{13}$$

On the other hand, from (11), it follows that

$$e^{-z_i(x)} = \frac{f_i(x)}{b_i v(x)}, \ i \in \{1, ..., p\}. \tag{14}$$

By substitution in (13), it implies that

$$v(x) = \sum_{i=1}^{p} f_i(x). \tag{15}$$

Now, from (11), (14) and (15), we deduce that

$$\sum_{i=1}^{p} z_i(x) f_i(x) = \sum_{i=1}^{p} (\ln b_i) f_i(x) + \left(\sum_{i=1}^{p} f_i(x)\right) \ln\left(\sum_{i=1}^{p} f_i(x)\right) - \sum_{i=1}^{p} f_i(x) \ln f_i(x). \tag{16}$$

Since $(z_1(x), ..., z_p(x)) \in Z$ is an optimal solution of the problem (9), the left-hand-side of the equality (16) coincides with $h(x)$. The right-hand-side of the equality (16) is equal with $g(x)$, because $\ln b_i = a_i$, $i \in \{1, ..., p\}$. The proof is complete.

Remark 2. If $a_i > 0$ for all $i \in \{1, ..., p\}$, then the condition (8) is satisfied.

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