REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome 29, N° 1, 2000, pp. 57–73

SOME GENERAL QUESTIONS OF THE THEORY OF SINGULAR OPERATORS IN THE CASE OF A PIECEWISE LYAPUNOV CONTOUR

VASILE NYAGA

Abstract. The theory of singular integral equations and boundary problems for analytic functions with piecewise Hölder and piecewise continuous coefficients along Lyapunov curves has been developed rather completely. Many works are devoted to this theme, among which we mention only fundamental ones [1–6]. Further the theory of singular integral equations and boundary problems developed intensively in such directions as, for example, weakening conditions on the class of functions under consideration [7–8], on the coefficients of equations and boundary problems [9–11], the extension of admissible curves [12–16], the study of singular equations which do not satisfy the condition of Hausdorff normal solvability [17], of singular integral equations with non-diagonal singularities (with shift) [18–25], etc.

The present work is a survey of problems and results related to the influence of corner points of integration contour on various properties of singular operators. The main attention is paid to properties obtained by the author and which differ from the corresponding properties in the case of a Lyapunov contour. Note that the case of an unlimited contour has been considered in the author's work [26].

1. ON THE ESSENTIAL NORM OF SINGULAR OPERATORS

THEOREM 1.1. *Let* Γ *be a piecewise Lyapunov contour with a finite number of selfintersection points,* $t_1, t_2, \ldots, t_n$ *be some points on* Γ, $\beta_1, \beta_2, \ldots, \beta_n$ *be real numbers and*

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}. \tag{1.1}$$

The operator

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} \mathrm{d}\tau \quad (\tau \in \Gamma) \tag{1.2}$$

2000 AMS Subject Classification: 45G05.

is bounded in the space $L_p(\Gamma,\rho)$ if and only if the numbers β_k $(k = 1, 2, ..., n)$ satisfy the conditions $-1 < \beta_k < p - 1$ $(k = 1, ..., n)$.

The sufficient part of this assertion is proved for a Lyapunov contour in [5], for a piecewise Lyapunov contour in [12], for $\Gamma = (-\infty, \infty)$ in [27] and for an arbitrary unbounded contour in [28] (see also [29]).

In [7] it was shown that the norm $\|S_0\|_p$ in the space $L_p(\Gamma_0)$ $(\Gamma_0 = \{t : |t| = 1\})$ for $p = 2^n$ and $p = 2^n(2^n - 1)^{-1}$ is equal to $\nu(p)$, where

$$\nu(p) = \begin{cases} \text{ctan}\pi/2p & \text{if} \quad 2 \leqslant p \leqslant \infty, \\ \tan\pi/2p & \text{if} \quad 1 < p \leqslant 2. \end{cases}$$

After this result has been obtained, a series of works appeared where the norms of singular operators in various spaces were evaluated and calculated. In [30] the norm of the operator

$$(C\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(y)\,\text{ctan}\frac{y-t}{2}dy$$

was calculated. It turned out that $\|C\|_p = \nu(p)$ (in the space $L_p(0, 2\pi)$). This permitted to prove that $\|S_0\|_p = \nu(p)$ for any $p \in (1, \infty)$.

In the monograph [7] the estimate of the essential norm $|S_\Gamma|_{P,\rho} = \inf\limits_{T\in\mathfrak{S}} \|S_\Gamma + T\|_{L_P(\Gamma,\rho)}$ in the space $L_P(\Gamma,\rho)$ in the case of a Lyapunov contour was obtained. In the book by F. Zigmund ([31] p. 83) it is proved that $\|S_\mathbb{R}\|_p \leq \|C\|_p$ therefore $\|S_\mathbb{R}\|_p = \nu(p)$. Taking also in consideration the equality $\|S_\mathbb{R}\|_p = \|S_0\|_{p,|t-t_0|^{p-2}}$ (see [11]) and M. Riesz interpolation theorem, we obtain that if $\min(0, p-2) \leqslant \beta \leqslant \max(0, p-2)$, then

$$\|S_0\|_{p,|t-t_0|^\beta} = \|S_0\|_p.$$

The series of investigation was completed in [32].

For the first time the case of contour with corner points was considered in [14] and the case of contour with selfintersection was considered in [33].

Let Γ_α have one corner point with angle $\pi\alpha$ $(0 < \alpha \leqslant 1)$, then $|S_\alpha|_2 = \text{ctan}\theta(\alpha)/2$, $|P_\alpha|_2 = |Q_\alpha|_2 = (\sin\theta(\alpha))^{-1}$, where $S_\alpha = S_{\Gamma_\alpha}$ and

$$\text{(1.3)} \qquad \text{ctan}\theta(\alpha) = \frac{1}{2}\max_{-1\leqslant x\leqslant 1}\left|(1+x)\left(\frac{1-x}{1+x}\right)^{\alpha/2} - (1-x)\left(\frac{1+x}{1-x}\right)^{\alpha/2}\right|.$$

In particular $|S_{1/3}|_2 = \dfrac{1+\sqrt{5}}{2}$ and $|S_{1/2}|_2 = \sqrt{2}$.

THEOREM 1.2. *Let Γ be a piecewise Lyapunov contour with corner points $t_1, ..., t_n$ and $\rho(t) = \prod\limits_{k=1}^{n}|t - t_k|^{\beta_k}$ $(-1 < \beta_k < 1)$, then*

$$|S_\Gamma|_{L_2(\Gamma,\rho)} = \max_{1\leqslant k\leqslant n}|S_{\alpha_k}|_{L_2(\Gamma_{\alpha_k},|t|^{\beta_k})}.$$

Let $\min\limits_{1\leqslant k\leqslant n}(\alpha_1, ..., \alpha_n) = \alpha_{k_0}$. If $\alpha_{k_0} = 1$, then

$$|S_\Gamma|_{L_2(\Gamma,\rho)} = \max_{1\leqslant k\leqslant n}\text{ctan}\pi\frac{1-|\beta_k|}{4}.$$

If $\rho(t) \equiv 1$, then $|S_\Gamma|_{L_2(\Gamma)} = \text{ctan}\dfrac{\theta(\alpha_{k_0})}{2}$. For the operators P_Γ and Q_Γ the equalities $|P_\Gamma| = |Q_\Gamma| = \left(|S_\Gamma|^2 + 1\right)/2|S_\Gamma|$.

In the space $L_P(\Gamma)$ the estimates

$$|S_\Gamma|_P \leqslant \begin{cases} \text{ctan}\dfrac{\theta(\alpha_{k_0})}{p} & \text{if } p = 2^n \\ \text{ctan}^t\dfrac{\theta(\alpha_{k_0})}{2^n}\cdot\text{ctan}^{1-t}\dfrac{\theta(\alpha_{k_0})}{2^{n+1}} & \text{if } 2^n < p < 2^{n+1} \end{cases},$$

where $t = (2^{n+1} - p)/p$, are valid.

Consider the case when Γ has selfintersection points. To formulate one result we introduce some notations, which will be also used further. Let Γ be a composite contour consisting of m simple piecewise Lyapunov closed curves $\gamma_1, ..., \gamma_m$, which have a point t_0 in common, $\rho(t) = \prod\limits_{k=0}^{n}|t - t_k|^{\beta_k}$ $(-1 < \beta_k < p - 1)$, $h_k = p(1 + \beta_k)^{-1}$ $(k = 0, 1, ..., n)$, $h_{n+1} = p$, $\overline{h}_k = \max\left(h_k, h_k(h_k - 1)^{-1}\right)$, $(k = 0, 1, ..., n+1)$ and $h = \max\left(\overline{h}_1, \overline{h}_2, ..., \overline{h}_{n+1}\right)$.

THEOREM 1.3. *For the essential norm of operators P_Γ, Q_Γ and S_Γ in the space $L_p(\Gamma,\rho)$ the following estimates are true:*

$$\text{(1.4)} \qquad \begin{aligned} &|P_\Gamma|_{p\rho}, |Q_\Gamma|_{p\rho} \geqslant \max\left((\sin\pi/h)^{-1}, (\sin\pi/m\overline{h}_0)^{-1}\right), \\ &|S_\Gamma|_{p\rho} \geqslant \max\left(\text{ctan}\pi/2h, \text{ctan}\pi/2m\overline{h}_0\right). \end{aligned}$$

These estimates are in concordance with the corresponding results from [32] and embrace all the cases of boundedness of operator S_Γ in $L_p(\Gamma,\rho)$. Remark that for one class of contours the above estimates are exact. So, if the tangents to Γ at selfintersection points are perpendicular and $\rho(t) \equiv 1$, then in (1.4) the equality sign takes place.

Let Γ be a simple closed piecewise Lyapunov contour which bounds a domain G_Γ^+, $\overline{\omega}$ be Riemann function mapping G_Γ^+ into $G_\Gamma^+ = \{z : |z| < 1\}$ and $t_1, ..., t_n$ be all the corner points of contour Γ with angles $\alpha_k\pi$ $(0 < \alpha_k \leqslant 1)$.

THEOREM 1.4. *Operator*

$$(K\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}\left(\frac{\overline{\omega}'(\tau)}{\overline{\omega}(\tau)-\overline{\omega}(t)} - \frac{1}{\tau - t}\right)\varphi(\tau)\,d\tau \tag{1.5}$$

is compact in the space $L_p(\Gamma,\rho)$, *if and only if* $\sum_{k=1}^{l}\alpha_k = l$.

THEOREM 1.5. *The operator* S_Γ^* *acting in the space* $L_q(\Gamma,\rho^{1-q})$ *has the form*

$$S_\Gamma^* = -VhSVhI, \tag{1.6}$$

where $(V\varphi)(t) = \overline{\varphi(t)}$ *and* h *is a piecewise Hölder function on* Γ.

THEOREM 1.6. *The operator* $S_\Gamma^* - S_\Gamma$ *is compact on the space* $L_2(\Gamma)$ *if and only if* $\sum_{k=1}^{l}\alpha_k = l$.

2. FACTORIZATION. NOETHER THEOREMS

$^{+}L_p^m(\Gamma,\rho) = P_\Gamma L_p^m(\Gamma,\rho)$; $^{-}L_p^m(\Gamma,\rho) = Q_\Gamma L_p^m(\Gamma,\rho) + \mathbb{C}$

Let Γ be a closed composite piecewise Lyapunov contour which bounds a domain G_Γ^+. By G_Γ^- we denote the domain which complements $G_\Gamma^+\cup\Gamma$ to the full plane. Assume that $0\in G_\Gamma^+$ and $\infty\in G_\Gamma^-$. Let $\mathfrak{B}^{m\times m}$ be the set of square matrices of order m with elements from $\mathfrak{B}$; $P_\Gamma = \|\delta_{ij}(I+S_\Gamma)/2\|_{i,j=1}^m$, $Q_\Gamma = I - P_\Gamma$; $^{+}L_p^m(\Gamma,\rho) = P_\Gamma L_p^m(\Gamma,\rho)$;

$^{-}L_p^m(\Gamma,\rho) = Q_\Gamma L_p^m(\Gamma,\rho) + \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers).

1°. Class $Fact_{p\rho}^m(\Gamma)$. *The generalized factorization of a matrix* $a\,(\in G\,L_\infty^{m\times m}(\Gamma))$ *with respect to contour* Γ *in the space* $L_p^m(\Gamma,\rho)$ *is defined (see [34], [7], [35]) its representation in the form*

$$a = a_- D a_+, \tag{2.1}$$

where $D = \|\delta_{jk}(t-z_0)^{\kappa_j}\|_1^m$, $j,i\leqslant 1$, $z_0\in G_\Gamma^+$; κ_j *are integers* ($\kappa_1\geqslant\kappa_2\geqslant\ldots\geqslant\kappa_m$), *and the factors* $a_\pm^{\pm}$ *satisfy the following conditions:*

i) $a_-\in {}^{-}L_p^{m\times m}(\Gamma,\rho)$, $a_+\in {}^{+}L_q^{m\times m}(\Gamma,\rho^{1-q})$, $a_-^{-1}\in {}^{-}L_q^{m\times m}(\Gamma,\rho^{1-q})$ *and* $a_+^{-1}\in {}^{+}L_p^{m\times m}(\Gamma,\rho)$; $(p^{-1}+q^{-1}=1)$;

ii) the operator $a_+^{-1}P_\Gamma a_+ I$ *is bounded in the space* $L_p^m(\Gamma,\rho)$.

The set of all matrix-functions $a\,(\in GL_p^{m\times m}(\Gamma))$ admitting generalized factorization with respect to contour Γ in the space $L_p^{m\times m}(\Gamma,\rho)$ will be denoted by $Fact_{p\rho}^m(\Gamma)$.

2°. Class $Nt_{p\rho}^m(\Gamma)$. *We regard measurable essentially bounded matrix-functions* $a(t)$ *as belonging to the Noether class (denoted by* $Fact_{p\rho}^m(\Gamma)$*), if the operator* $A = aP_\Gamma + Q_\Gamma$ *is Noetherian.*

THEOREM 2.1. *(see [34]).* $Nt_{p\rho}^m(\Gamma) = Fact_{p\rho}^m(\Gamma)$.

3°. The connection between $Fact_{p\rho}^m(\Gamma)$ **and** $Fact_p^m(\Gamma)$

Let h be a function from $Fact_p^m(\Gamma)$ such that

$$h = h_-\cdot h_+.$$

Denote $\mathfrak{B}_1 = L_p^m(\Gamma)$ and $\mathfrak{B}_2 = L_p^m(\Gamma,\rho\,|h_+|^{-p})$, $(1<p<\infty)$.

THEOREM 2.2. *There exist inversible operators* $B:\mathfrak{B}_2\to\mathfrak{B}_1$ *and* $C:\mathfrak{B}_1\to\mathfrak{B}_2$ *such that for any pair of matrix-functions* $a,b\in L_\infty^{m\times m}(\Gamma)$ *the equality*

$$B(aP_\Gamma + bQ_\Gamma)C = haP_\Gamma + bQ_\Gamma \tag{2.2}$$

holds.

The proof is contained in [36].

COROLLARY 2.1. *The operator* $A = aP_\Gamma + bQ_\Gamma$ $(a,b\in L_\infty^{m\times m}(\Gamma))$ *is Noetherian in the space* $L_p^m(\Gamma,|h_+|^{-p})$ *if and only if the operator* $A_h = haP + bQ_\Gamma$ *possesses the same property in the space* $L_p^m(\Gamma)$. *Then* $\dim\ker A|_{L_p^m(\Gamma,|h_+|^{-p})} = \dim\ker A_h|_{L_p^m(\Gamma)}$ *and* $\dim\ker A^*|_{L_q^m(\Gamma,|h_+|^{-p(1-q)})} = \dim\ker A_h^*|_{L_q^m(\Gamma)}$.

COROLLARY 2.2. $a\in Fact_{p\rho|h_+|^{-p}}^m(\Gamma)\Longleftrightarrow ah\in Fact_{p\rho}^m(\Gamma)$.

Show (see [37–38]), that Theorem 2.2 permits to reduce the investigation of the operator $A = aP_\Gamma + bQ_\Gamma$ in the space $\rho(t) = \prod_{k=1}^{n}|t-t_k|^{\beta_k}$ to the investigation of some singular operator in the space $L_p^m(\Gamma)$ (without weight). For simplicity we assume that Γ consists of ν closed curves $\Gamma_1,\ldots,\Gamma_\nu$ having a point t_0 in common and

$$\rho(t) = \prod_{k=0}^{n}|t-t_k|^{\beta_k},\ (-1<\beta_k<p-1).$$

Denote by Γ_{i_k} $(1\leqslant i_k\leqslant\nu)$ the curve containing point t_k and set

$$h_k(t) = \begin{cases}(t-z_k)^{-\frac{\beta_k}{p}} & \text{for}\quad t\in\Gamma_{i_k},\\ 1 & \text{for}\quad t\in\Gamma\backslash\Gamma_{i_k},\end{cases}$$

where z_k is a point of the domain $G_{i_k}^+$, bounded by the curve Γ_{jk}, a $(t-z_k)^{-\frac{\beta_k}{p}}$ is a branch of this function continuous at any point $t\in\Gamma_{i_k}$ different from t_k.

Let $\varpi_1, \ldots, \varpi_\nu$ be some points belonging, respectively, to the domains $G_1^+, \ldots, G_\nu^+$, $\sigma_1, \ldots, \sigma_\nu$ be some real numbers and $\widetilde{h}_k(z)$ be a fixed branch of the function $(z-\varpi_k)^{\sigma_k}$ defined on the complex plane $\mathbb{C}$ with a cut which joins z_k and ∞ and intersects the contour Γ in one point t_0. The functions $\widetilde{h}_k(t)$, $(k=1, \ldots, \nu)$ are continuous at any point $t \in \Gamma$, perhaps except the point t_0 : $\widetilde{h}_k(t_0 \pm 0) \neq 0$ and

$$\widetilde{h}_k(t_0-0)/\widetilde{h}_k(t_0+0) = \exp(2\pi i\sigma_k), \tag{2.3}$$

where the numbers $\widetilde{h}_k(t_0-0)$ and $\widetilde{h}_k(t_0+0)$ are determined by the equalities 3.1 of Chap. X of the work [7] (see also [16]). By $h(t)$ we denote the product

$$h(t) = h_1(t) \cdot \ldots \cdot h_n(t)\, \widetilde{h}_1(t) \cdot \ldots \cdot \widetilde{h}_\nu(t). \tag{2.4}$$

THEOREM 2.3. *(see [37]). Let* $\sum_{k=1}^{\nu} \sigma_k = -\frac{\beta_0}{p}$. *Then*

$$a \in Fact_{p\rho}^m(\Gamma) \iff ah \in Fact_p^m(\Gamma).$$

COROLLARY 2.3. *Let* $A = aP_\Gamma + bQ_\Gamma$, *h be a function determined by the equality (2.6) and* $A_h = ahP_\Gamma + bQ_\Gamma$. *Then* $A \in Nt_{p\rho}^m(\Gamma) \iff A_h \in Nt_p^m(\Gamma)$. *By this* $Ind\, A = Ind\, A_h$.

4°. Class $M_\rho(\Gamma)$ (see [39]). In this and next items we assume that Γ consists of two curves Γ_1 and Γ_2, having one common point t_0, and besides the tangents to Γ at this point are perpendicular and $\rho(t)$ is the function determined by the equality (2.4).

Let τ_0 be a point on Γ different from t_0. Denote by $\Lambda(\tau_0)$ the closed halfplane which does not contain the origin. By $\Delta(\tau_0)$ we denote the angle with vertex at the origin and value of $\pi/2$. To the class $M_\rho(\Gamma)$ refer essentially bounded measurable functions $a(t)$ satisfying the conditions:

(i) essinf $|a(t)| > 0$; $t \in \Gamma$;

(ii) for any point $t \in \Gamma \backslash \{t_0\}$ there exist a neighbourhood $u(t)$ $(\subset \Gamma \backslash \{t_0\})$ of the point τ and a pair of functions $g_\tau^\pm(t)$ such that $(g_\tau^+(t))^{\pm 1} \in L_\infty^+(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in L_\infty^-(\Gamma)$ and the range of the function $g_\tau^+(t)\, h(t)\, a(t)\, g_\tau^-(t)$ at $t \in u(\tau)$ is contained inside $\Lambda(\tau)$.

(iii) for the point t_0 either there exist a neighbourhood $u(t_0)$ and functions $g_\tau^\pm(t)$ such that $(g_0^\pm(t))^{\pm 1} \in L_\infty^\pm(\Gamma)$ and the range of the function $g_0^+(t)\, h(t)\, a(t)\, g_0^-(t)$ at $t \in u(t_0)$ is contained inside $\Lambda(t_0)$, or there exist finite limits $a(t_0 \pm 0)$ and $h_{t_0}(t_0-0)\, a(t_0-0)/h_{t_0}(t_0+0)\, a(t_0+0) \notin (-\infty, 0)$.

THEOREM 2.4. $M_\rho(\Gamma) \subset Fact_{2\rho}(\Gamma)$.

COROLLARY 2.4. *Let* $a \in M_\rho(\Gamma)$. *Then the operator* $A = aP + Q$ *is Noetherian in the space* $L_2(\Gamma, \rho)$.

COROLLARY 2.5. $M_\rho(\Gamma) \cap PC(\Gamma) = Fact(\Gamma) \cap PC(\Gamma)$, *where* $PC(\Gamma)$ *is a set of all piecewise continuous functions on* Γ.

Note that if Γ is a simple closed Lyapunov contour and $\rho(t) \equiv 1$, then the class $M_1(\Gamma)$ coincides with the class $\widetilde{A}(2, \Gamma)$ introduced by I. B. Simonenko (see [40]). In this case, as known (see [40]), $aP + bQ$ is Noetherian if and only if $a \in \widetilde{A}(2, \Gamma)$ $(= M_1(\Gamma))$. From this and Theorem 2.4 follows

THEOREM 2.5. $M_\rho(\Gamma) = Nt_{2\rho}(\Gamma)$.

5°. Class $M_\rho^m(\Gamma)$ (see [36]). To class $M_\rho^m(\Gamma)$ we refer matrix-functions $a(t) = \|a_{jk}(t)\|_{j,k}^m$ $(t \in \Gamma)$ of order m with elements $a_{jk} \in L_\infty(\Gamma)$ satisfying the conditions:

(i) essinf $|\det a(t)| > 0$, $(t \in \Gamma)$;

(ii) for any point $\tau \in \Gamma$ except, perhaps, a finite number of points t_0, τ_k $(k = 1, \ldots, l)$, there exist a neighborhood $u(t)$ $(\subset \Gamma)$ of the point τ and a pair of matrix-functions $g_\tau^\pm$ such that $(g_\tau^+(t))^{\pm 1} \in {}^+L_\infty^{m\times m}(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in {}^-L_\infty^{m\times m}(\Gamma)$ and for any $t \in u(\tau)$

$$\mathrm{Re}(g_\tau^+(t) h_\tau(t) a(t) g_\tau^-(t)) \geq \sigma(t) > c(\tau)\cos\theta(\tau)$$

where $\mathrm{Re}B = (B + B^*)/2$, B^* is the matrix conjugate to B, $\theta(\tau)$ is the function from Theorem 1.2 and $c(\tau)$ is the norm of the operator $h_\tau a I$ in the space $L_2^m(u(\tau))$. Note that $c(\tau)$ coincides with $\sup_{t \in u(\tau)} s_1(h_\tau(t)\, a(t))$, where $s_1(h_\tau a)$ is the greatest eigenvalue of the matrix $(h_\tau a a^* h_\tau^*)^{1/2}$;

(iii) there exist finite limits a_{2j} (a_{2j-1}) of matrix-function $a(t)$ as t tends to t_0 along arc Γ_j $(j = 1, 2)$ directed to point t_0 (from point t_0) and the spectrum of matrix $e^{-\pi i\beta_0} a_4 a_3^{-1} a_2 a_1^{-1}$ does not intersect the negative semi-axis $\mathbb{R}^-$;

(iv) at points τ_k there exist finite limits on the left and on the right $a(\tau_k - 0)$ and $a(\tau_k + 0)$ of the matrix $a(t)$ and the spectrum of matrix

$$e^{-\pi i\beta(\tau_k)} a^{-1}(\tau_k + 0) a(\tau_k - 0)$$

does not intersect the negative semi-axis $\mathbb{R}^-$;

It easy to see that if all the points τ_k $(k = 1, \ldots, l)$ at which there exist limits $a(\tau_k \pm 0)$ are ordinary (see [1], p. 16), then the conditions (iv) are equivalent to conditions (ii). This can be deduced also from Lemma 2.1 of the work [15].

THEOREM 2.6. *Let the matrix-function a belong to the set* $M_\rho^m(\Gamma)$, *then the operator* $A = aP + Q$ *is Noetherian in the space* $L_2^m(\Gamma, \rho)$.

Remark that for Γ being a closed Lyapunov contour and $\rho(t) \equiv 1$ this theorem was proved by I. B. Simonenko (see [41], Theorem 8), and the set

$M_1^m(\Gamma)$ coincides with class $\widetilde{A}^m(2,\Gamma)$ introduced in [47]. For Γ being a simple Lyapunov contour Theorem 2.6 is contained in the work by N. Ia. Krupnik [42]. Note also that from Theorem 2.6 follows that if $a, b \in L_\infty^{m\times m}(\Gamma)$, essinf$|\det b(t)| > 0$ $(t \in \Gamma)$ and $b^{-1}a \in M_\rho^m(\Gamma)$, then the operator $aP + bQ$ is Noetherian in the space $L_2^m(\Gamma,\rho)$.

COROLLARY 2.6. $M_\rho^m(\Gamma) \subset Fact_{2\rho}^m(\Gamma)$.

Theorems 2.5 and 2.6 are transferred, with corresponding changes, to the case of an unclosed contour.

6°. Class $G_{\delta\rho}(\Gamma)$. Let Γ be a closed piecewise Lyapunov contour with corners $t_1, ..., t_n$ and $\rho(t)$ be the weight determined by the equality (1.1). By $G_{\delta\rho}(\Gamma)$ denote the set of all matrices a of $L_\infty^{m\times m}(\Gamma)$ satisfying the following conditions:

(i) essinf $|\det a(t)| > 0$, $(t \in \Gamma)$;

(ii) for each point $\tau \in \Gamma\backslash\{t_1, ..., t_s\}$ $(s \geqslant n)$, there exist a neighbourhood $u(\tau)$ $(\subset \Gamma)$, there exist a neighbourhood $u(\tau)(\subset\Gamma)$ of point τ and a pair of matrix-functions $g_\tau^\pm$ such that $(g_\tau^+(t))^{\pm 1} \in ({}^+L_\infty^{m\times m})(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in {}^-L_\infty^{m\times m}(\Gamma)$ and for any $t \in u(\tau)$, the matrix $g_\tau^+ a g_\tau^-$ is unitary and $\sigma(g_\tau^- a g_\tau^+) \subset \Lambda_\tau(\delta)$, where $\Lambda(\delta)$ $(0 < \delta < 2\pi)$ denoted angle with vertex of the point $z = 0$ and value less than δ.

(iii) for the points t_k $(k = 1, ..., n)$ there exists finite limits $a(t_k \pm 0)$ and $\det(f_k(\mu)a(t_k+0) + (1 - f_k(\mu))a(t_k - 0)) \neq 0$ $(0 \leqslant \mu \leqslant 1)$, where

$$f_k(\mu) = \begin{cases} \dfrac{\sin\theta_k\mu\exp(i\theta_k\mu)}{\sin\theta_k\exp(i\theta_k)}, & \theta_k = \pi - 2\pi(1+\beta_k)/p, \quad \text{if } \theta_k \neq 0, \\ \mu & \text{if } \theta_k = 0 \end{cases}$$

(iv) there exist a neighbourhood $u(t_k)$ $(k = n+1, ..., s)$ and a pair of matrix functions g_k^+, g_k^- such that $(g_k^+(t))^{\pm 1} \in L_\infty^{m\times m}(\Gamma)$, $(g_k^-(t))^{\pm 1} \in L_\infty^{m\times m}(\Gamma)$ for any $t \in u(t_k)$, the matrix $g_k^+ a g_k^-$ is unitary and

$$\sigma(g_k^- a g_k^+) \subset \Lambda_k(\delta) \quad (t \in u^+(t_k)), \ \sigma\left(g_k^- a g_k^+ e^{\frac{2\pi i\beta_k}{p}}\right) \subset \Lambda_k(\delta) \quad (t \in u^-(t_k)),$$

where $u^+(t_k) = \{t \in u(t_k), t \succ t_k\}$ and $u^+(t_k) = \{t \in u(t_k), t \prec t_k\}$.

THEOREM 2.7. *(see [15]). Let $a \in G_{\delta\rho}(\Gamma)$, where*

$$\tan\delta/2 = \min\left(m^{\frac{2-p}{2p}}\tan\frac{\pi}{2p}, \ m^{\frac{p-2}{2p}}\text{ctan}\frac{\pi}{2p}\right).$$

then the operator $aP + Q$ is Noetherian in the space $L_p^m(\Gamma,\rho)$.

COROLLARY 2.7. $G_{p\rho}(\Gamma) \subset Fact_{p\rho}^m(\Gamma)$.

3. THE DEPENDENCE OF NOETHERIAN PROPERTY OF SINGULAR INTEGRAL OPERATORS WITH SHIFT AND CONJUGATION ON THE EXISTENCE OF A CORNER POINTS ON CONTOUR

1°. ***Singular operators with shift.*** Let Γ be a closed piecewise Lyapunov contour, $\nu : \Gamma \to \Gamma$, $(V\varphi)(t) = \varphi(\nu(t))$. In the space $L_p(\Gamma)$ consider a singular integral operator with shift $\nu(t)$ of the form $A = a(t)I + b(t)S + (c(t)I + d(t)S)V$, where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are bounded measurable functions on Γ. Assume that the mapping ν satisfies the following conditions:

(i) $\nu(\nu(t)) \equiv 1$;

(ii) the derivative $\nu'(t)$ has on Γ a finite number of discontinuity points of the first kind and on arcs l_k, joining discontinuity points it satisfies the Hölder conditions;

(iii) $\nu(t \pm 0) \neq 0$ $(t \in \Gamma)$.

Together with the operator A of the form (3.1) consider also the operator $\widetilde{A}$ determined in the space $L_p^2(\Gamma) = L_p(\Gamma) \times L_p(\Gamma)$ by the equality

$$(3.1) \qquad \widetilde{A} = \left\| \begin{matrix} aI + bS & cI + dS \\ \widetilde{c}I + \varepsilon\widetilde{d}S & \widetilde{a}I + \varepsilon\widetilde{b}S \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ \widetilde{d}(V_sV - \varepsilon S) & \widetilde{b}(V_sV - \varepsilon S) \end{matrix} \right\| = \widetilde{A}_0 + R,$$

where $\widetilde{f}(t) = f(\nu(t_1))$ and $\varepsilon = 1$ $(\varepsilon = -1)$, if the mapping ν preserves (reverses) orientation on Γ. As is known [4], if a, b, c and d are continuous functions and $\nu'(t) \in H(\Gamma)$, then the operator R is compact in L_p^2 and the following theorem is true.

THEOREM 3.1. $A \in Nt_p(\Gamma) \Longleftrightarrow \widetilde{A}_0 \in Nt_p^2(\Gamma)$, *by this* $Ind\,A = \frac{1}{2}Ind\widetilde{A}_0$.

Show (see [43]), that the assertion ceases to be true if Γ has corner points. In such case, usually, the derivative $\nu'(t)$ has on Γ discontinuity point, and it turns out that if the operator A is Noetherian, then the operator $\widetilde{A}_0$ is also Noetherian but the converse does not hold.

THEOREM 3.2. *If the operator A $(a, b, c, d \in C(\Gamma))$ is Noetherian in the space $L_p(\Gamma)$, then the operator $\widetilde{A}_0$ is also Noetherian in the space $L_p^2(\Gamma)$.*

Indeed, the operator $\widetilde{A}_0$ is Noetherian if and only if

$$\Delta_1(t) = (a(t) + b(t))\left(\widetilde{a}(t) + \varepsilon\widetilde{b}(t)\right) - (c(t) + d(t))\left(\widetilde{c}(t) + \varepsilon\widetilde{d}(t)(t)\right) \neq 0$$

and

$$\Delta_2(t) = (a(t) - b(t))\left(\widetilde{a}(t) - \varepsilon\widetilde{b}(t)\right) - (c(t) - d(t))\left(\widetilde{c}(t) - \varepsilon\widetilde{d}(t)(t)\right) \neq 0$$

for any $t \in \Gamma$. Let the operator A be Noetherian, then the determinant of its symbol (see [22]) is not equal to zero: $\det A(t_1, \xi) \neq 0$. One can check directly that

$$\det A(t, -\infty) \cdot \det A(t, +\infty) = \Delta_1(t) \cdot \Delta_2(t).$$

Therefore, the operator $\widetilde{A}_0$ is Noetherian in $L_p^2(\Gamma)$. The theorem is proved.

The following example shows that the converse of Theorem 3.2 is not true. Let ν reverse orientation on Γ and the corner point t_0 $(\in \Gamma)$ with the angle θ $(0 < \theta < \pi)$ be a fixed point of the mapping $\nu : \nu(t_0) = t_0$. In this case it is easy to see that the derivative $\nu'(t)$ is discontinuous at the point t_0 and $\nu'(t+0) = \exp(\mathrm{i}\theta + \sigma)$, $\nu'(t-0) = \exp(\mathrm{i}\theta - \sigma)$, where σ is a real number. Consider the operator

$$A = I + \delta SV,$$

where δ is complex number. The operator $\widetilde{A}$ has the form

$$\widetilde{A} = \left\| \begin{matrix} I & \delta S \\ -\delta S & I \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ \delta(VSV + S) & 0 \end{matrix} \right\| \neq \widetilde{A}_0 + R.$$

If $\delta \neq \pm\mathrm{i}$, the operator $\widetilde{A}_0$ is Noetherian. Let $A(t, \xi)$ be symbol of the operator A at the point t_0. One can check directly that

$$\det A(t_0, \xi) = \delta^2 + 2(\alpha + \beta)\delta + 1, \text{where}$$

$$\alpha = \frac{\exp\left[(2\pi - \theta - \mathrm{i}\sigma)\left(\xi + \frac{\mathrm{i}}{\pi}\right)\right]}{\exp\left(\xi + \frac{\mathrm{i}}{\pi}\right) - 1} \text{ and } \beta = \frac{\exp\left[(\theta + \mathrm{i}\sigma)\left(\xi + \frac{\mathrm{i}}{\pi}\right)\right]}{\exp\left(\xi + \frac{\mathrm{i}}{\pi}\right) - 1}$$

From this, due to Theorem 1.1 from [22], it follows that for any $\delta = -(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 1}$ the operator A is Noetherian in the space $L_p(\Gamma)$. Thus, the conditions for operator A being Noetherian depend on angle θ.

COROLLARY 3.1. *Let $\nu'(t) \notin H(\Gamma)$, then the operator $VSV - \varepsilon S$ is not compact in $L_p(\Gamma)$.*

COROLLARY 3.2. *If the operator A, determined by the equality 3.1 is Noetherian, then the operators $\widetilde{A}$ and $\widetilde{A}_0$ determined by the equality 3.2 are also Noetherian and $\operatorname{Ind} A = \operatorname{Ind} \widetilde{A}_0$.*

COROLLARY 3.3. *If the operator $\widetilde{A}$ is Noetherian, then $\widetilde{A}_0$ is also Noetherian. The converse is not true, in general.*

Note that the corresponding example of non-Noetherian operator A for which the operator $\widetilde{A}_0$ is Noetherian can be given also in the case when ν preserves the orientation of contour Γ.

2°. *Singular operators with conjugation.* Singular integral operators with conjugation have the form

$$A = aP + bQ + (cP + sQ)V,$$

where $a, b, c, d \in PC_m(\Gamma)$, $P = (I + S)/2$, $Q = I - P$, $(V\varphi)(t) = \overline{\varphi(t)}$ and Γ is a closed piecewise Lyapunov contour.

By constructing Nether theory of operator A in the monograph [4] an essential role was played by the fact that if at each point of contour Γ the Lyapunov condition is satisfied, then the operator $VSV + S$ is compact in the space $L_p(\Gamma, \rho)$. In this case the operator A is (see[4]) Noetherian if and only if the operator

$$A_V = \left\| \begin{matrix} a & c \\ \overline{d} & \overline{b} \end{matrix} \right\| P + \left\| \begin{matrix} b & d \\ \overline{c} & \overline{a} \end{matrix} \right\| Q$$

possesses the same property in the space $L_p^2(\Gamma, \rho) = L_p(\Gamma, \rho) \times L_p(\Gamma, \rho)$.

It is quite different if the contour Γ has corner points. It turns out that in this case the operator $VSV + S$ is not compact in $L_p(\Gamma, \rho)$ and if A is Noetherian, then A_V is also Noetherian but the converse ceases to be true. It is these facts which constitute significant difference between a piecewise Lyapunov contour and a Lyapunov contour.

THEOREM 3.3. *The operator*

$$(VSV + S)\varphi = -\frac{1}{\pi\mathrm{i}} \int_\Gamma \frac{\overline{\varphi(\tau)\, d\tau}}{\overline{\tau} - \overline{t}} + \frac{1}{\pi\mathrm{i}} \int_\Gamma \frac{\varphi(\tau)\, d\tau}{\tau - t}$$

is compact in the space $L_p(\Gamma, \rho)$ if and only if Γ is a Lyapunov contour.

The sufficient part of this assertion has been proved in [4]. Prove the necessity. Let $VSV + S$ be compact, then the operator $A_\lambda = VSV + S - \lambda I$ is Noetherian for all $\lambda \in C \setminus \{0\}$. Therefore, due to Theorem 1 of work [48], $\det A_\lambda(t_k, \xi) \neq 0$ for all $k+1, \ldots, s$ and $-\infty < \xi < \infty$, where t_k $(k = 1, \ldots, s)$ are all corner points of contour Γ. From this we obtain that

$$\frac{z_k^{2\pi - \theta_k} - z_k^{\theta_k}}{z_k^{2\pi} - 1} \equiv 0 \left(z_k = \exp\left(\xi + \mathrm{i}\frac{1 + \beta_k}{p}\right)\right).$$

The last is possible only for $\theta_k = \pi$. The theorem is proved.

The condition for operator A being Noetherian, unlike singlar operators not containing the operator V (i.e.$aP + bQ$) depends essentially on contour. For example, the operator $A = (1 + \sqrt{2})P + (1 - \sqrt{2})Q + V$ is Noetherian in all spaces $L_p(\Gamma, \rho)$ if Γ is a Lyapunov contour and is not Noetherian in $L_2(\Gamma)$ if Γ has one corner point with angle $\pi/2$.

3°. *Generalized Riemann problem.* Consider the generalized boundary Riemann problem: to find analytically representable by Cauchy integral in F_Γ^+ and F_Γ^- functions $\Phi^+(z)$ and $\Phi^-(z)$ limit values of which on Γ belong to the space $L_p(\Gamma,\rho)$ and satisfy the conditions

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t)\overline{\Phi^-}(t) + c(t),$$

where $a(t)$, $b(t)$ are defined on Γ continuous functions and $c(t) \in L_p(\Gamma,\rho)$.

Noether theory of this problem in the case of a Lyapunov contour has been constructed in the works [45], [46], [4]. In particular, in these works it is established that a necessary and sufficient condition for the problem being Noetherian is that the inequality $|a(t)| > 0$ should be satisfied for all $t \in \Gamma$. In the case of a piecewise Lyapunov contour the following theorem is true (see [47], [49]).

THEOREM 3.4. *In order that the generalized boundary Riemann problem in $L_p(\Gamma,\rho)$ be Noetherian it is necessary and sufficient that the following conditions be satisfied:*

(i) $|a(t)| > 0 \quad (t \in \Gamma)$;

(ii) $|a(t_k)|^2 - |b(t_k)|^2 \left(\dfrac{z_k^{2\pi-\theta_k} - z_k^{\theta_k}}{z_k^{2\pi}-1}\right) \neq 0$ *for all* $k = 1, \ldots, n$, *where* $z_k = \exp\left(\xi + i\dfrac{1+\beta_k}{p}\right)$, $-\infty \le \xi \le \infty$, $\theta_k = \theta(t_k)$ *and* $\beta_k = \beta(t_k)$.

This, in the case of a piecewise Lyapunov contour the Noetherian property of Riemann problem depends not only on the coefficient $a(t)$, as it was in the case of a Lyapunov contour, but also on the coefficient $b(t)$.

The results of this section can be extended, without essential changes, to the case when Γ consists of a finite number of closed piecewise Lyapunov curves without common points.

4. PERTURBATION OF SINGULAR INTEGRAL OPERATORS

In this section we will show that the Noetherian property of operators $aP+bQ$ is stable under perturbation by some not compact operators. Remark that analogous questions have also been studied in [53] and [54]. For simplicity assume that $\Gamma\,(=\{t : |t| = 1\})$ is a unit circle. Let α_k $(k = 1, \ldots, s)$ be some complex numbers. Introduce the following notations:

$$\Gamma = \{\xi : \xi = t - \alpha_k,\ t \in \Gamma\} \text{ and } \widetilde{\Gamma}_k = \{\xi : \xi = t + \alpha_k,\ t \in \Gamma\}.$$

Assume that $\Gamma_j \cap \Gamma \cap \widetilde{\Gamma}_k = \emptyset \quad (j,k = 1, \ldots, s)$.

THEOREM 4.1. *Let a, b, $c_k \in L_\infty(\Gamma)$ $(k = 1, \ldots, s)$. In order that the operator*

$$(4.1)\quad (A\varphi)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \sum_{k=1}^{s} c_k(t)\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t - \alpha_k}d\tau$$

be Noetherian in the space $L_p(\Gamma,\rho)$ it is necessary and sufficient that the operator

$$(4.2)\quad (A\varphi)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$$

possesses the same property. If the operator A_0 is Noetherian, then $IndA = IndA_0$.

In particular, if $a, b \in C(\Gamma)$, the condition $a^2(t) - b^2(t) \neq 0\ (t \in \Gamma)$ is a necessary and sufficient condition for the operator A being Noetherian and

$$IndA = Ind\frac{a(t)+b(t)}{a(t)-b(t)}.$$

In the general case $(a, b \in L_\infty(\Gamma))$ one can apply the criteria from Section 2 to the operator A. Note that the operator A_0 is (see [1]) the characteristic part of the operator A.

It turns out that the operators with kernels $(\tau - t - \alpha_k)^{-1}$ are not, in general, compact in the space $L_p(\Gamma,\rho)$ (see [51]). Proof of theorem 4.1 is based on a series of properties of operators with kernels $(\tau - t - \alpha_k)^{-1}$ and their compositions with operator S and operators aI. The conditions $\Gamma_k \cap \Gamma \cap \widetilde{\Gamma}_j = \emptyset$ is essential. For example

$$(4.3)\quad (A\varphi)(t) = \lambda\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t - 1}d\tau + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t + 1}d\tau$$

$\lambda \in \mathbb{C}$ is a Noetherian in $L_2(\Gamma)$ for $\lambda = 2i$, whereas $(A_0\varphi)(t) = \lambda\varphi(t)$ is invertible for all $\lambda \neq 0$.

DEFINITION 4.1. *The subset $(t_1,t_2), (t_2,t_3), \ldots, (t_m,t_1)$ of the set $\Gamma \times \Gamma$ is called m-link if $t_j \neq t_k$ for $j \neq k$ [53].*

DEFINITION 4.2. *The set $M \subset \Gamma \times \Gamma$ is called admissible if there exists a neighborhood of this set which does not contain m-links for any m.*

DEFINITION 4.3. *Let*

$$(4.4)\quad (H\varphi)(t) = \int_\Gamma h(\tau,t)\varphi(\tau)d\tau \quad (t \in \Gamma)$$

We say that the essential singularity of the kernel $h(\tau,t)$ is contained in the set M if the integral operators with the kernel

$$\widetilde{h}(\tau,t) = \begin{cases} 0 & \text{in a neighborhood of the set } M, \\ h(\tau,t) & \text{at other points of } \Gamma \times \Gamma, \end{cases}$$

is compact.

In [53] the following assertion is proved.

THEOREM 4.2. *Let the essential singularity of the kernel* $h(\tau, t)$ *of the integral operator (4.4) be contained in an admissible set* M.

If $A_0 = aI + bS \in N_{p\rho}(\Gamma) \Rightarrow A = aI + bS + H \in N_{p\rho}(\Gamma)$ *and* $IndA_0 = IndA$.

Denote by M the set of pairs $(\tau, t) \in \Gamma \times \Gamma$ for which $\tau - t - \alpha_k = 0$ $(k = 1, ..., s)$. Assume that the numbers α_k are such that $M \neq \emptyset$. Then the set M consists of finite number of points (τ_1, t_1), ..., (τ_N, t_N) and the operator

$$(K\varphi)(t) = \sum_{k=1}^{S} C_k(t) \frac{1}{\pi} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t - \alpha_k} d\tau \tag{4.5}$$

is not compact in the space $L_p(\Gamma, \rho)$. From Theorem 4.2 one can deduce (see [53]) the following proposition.

THEOREM 4.3. *Let the set* M *do not contain m-links* $(m = 1, ..., N)$. *In order that the operator* $A = aI + bS + K$ $(a, b \in PC(\Gamma))$ *be Noetherian in the space* $L_p(\Gamma, \rho)$ *it is necessary and sufficient that the operator* $A_0 = aI + bS$ *possess the same property. If the operator* A_0 *is Noetherian, then* $IndA = IndA_0$.

The above example proves that the restrictions on the set of singularities are, in some sense, exact. In this connection the question of the necessity of conditions of Theorem 4.3 arises naturally.

To be more exact, whether there exists an operator of the form (4.6) satisfying the following conditions: 1) the set M contains an m-link; 2) $|c_k(t)| \geq \delta > 0$ $(k = 1, ..., s)$ on the set M; 3) the operator $A = aI + bS + K \in Nt_{p\rho}(\Gamma) \Leftrightarrow A_0 = aI + bS + K \in Nt_{p\rho}(\Gamma)$.

Such operators exist (see [52]). As operator K we take the operator acting by the rule

$$(K\varphi)(t) = \frac{1}{i\pi} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t - 2} d\tau + \frac{1}{i\pi} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t + 2} d\tau. \tag{4.6}$$

Note that the operator K is not (see [51]) compact in the space $L_p(\Gamma, \rho)$. The set M corresponding to the operator K consists of two points $(-1, 1)$ and $(1, -1)$, forming two links. Denote by $\mathfrak{N}$ the set of all functions from $PC(\Gamma)$ continuous in some neighborhood of the points $\tau = \pm 1$.

THEOREM 4.4. *Let* $a, b \in \mathfrak{N}$

$$A = aI + bS + K \in Nt_{p\rho}(\Gamma) \Leftrightarrow A_0 = aI + bS \in Nt_{p\rho}(\Gamma).$$

If $A_0 \in Nt_{p\rho}(\Gamma)$, *then* $IndA = IndA_0$.

In conclusion remark that the results of this section can be transferred to case where Γ is an arbitrary piecewise Lyapunov contour which has no straight-line parts as well as to operators of the form (4.1) with matrix coefficients.

REFERENCES

[1] MUHELISHVILI, N. I., *Singulearne integraline pravnenia*, Moskva, Nauka, 1968.

[2] GAHOV, F. D., *Craeve zadaci*, Moskva, Nauka, 1977.

[3] VEKUA, I. P., *Sistem singulearnh integralinh uravnenii i necotore granicine zadaci*, Moskva, Nauka, 1970.

[4] LITVINCIUC, G. S., *Craeve zadaci i singulearne integraline uravnenia so sdvigom M*, 1977.

[5] HVEDELIDZE, B. V., *Lineine razvrne granicine zadaci teorii funcţii, singulearne integraline uravnenia i necotore ih prilojenia*, Trud Tbilisscogo matematicescogo instituta AN Gruz SSR, **23**, pp. 3–190, 1956.

[6] MIKHLIN, S. G., *Singulearne integraline uravnenia*, UMN, **3**, no. 3, pp. 29–112, 1948.

[7] GOHBERG, I. Ţ., KRUPNIK, N. I., *Vvedenie v teoriu odnomernh singulearnh integralinh operatorov*, Chişinău, Ştiinţa, 426 pp, 1973.

[8] DUDUCEAVA, R. V., *O singulearnh integralinh operatorah v ghiolderovscom prostranstve s vesom*, DAN SSSR, **191**, nr. 1, pp. 16–19, 1970.

[9] SIMONENCO, I. B., *Craevia zadacia Riemanna s neprervnmi coeffiţientami*, DAN SSSR **124**, nr. 2, pp. 278–281, 1959.

[10] DANILIUK, I. I., *O zadace Rimanna s izmerimmi coeffiţientami*, Sibirschii matematiceskii jurnal, **1**, nr. 2, pp. 171–197, 1960.

[11] KRUPNIK, N. I., NEAGA V. I., *O singulearnh operatorah v prostranstvah* L_p *s vesom*, Matem. Issled., Chişinău, Ştiinţa, **9**, nr. 3, pp. 206–209, 1974.

[12] GORDADZE, Ă. G., *O singulearnh integralah s iadrom Cauchy*, Soobşcenie AN Gruz. SSR, **37**, nr. 3, pp. 521–526, 1965.

[13] DANILIUC, I. I., *Neregulearne granicine zadaci na ploscosti*, Moskva, Nauka, 1975.

[14] NEAGA, V. I., *O simvole singulearnh integralinh operatorov v sluciae cusocinoleapunovscogo contura*, Matem. Issled., Chişinău, Ştiinţa, **9**, nr. 2, pp. 109–125, 1974.

[15] KRUPNIK, N. I., NEAGA V. I., *O singulearnh integralinh operatorah v sluciae negladcogo contura*, Matem. Issled., Chişinău, Ştiinţa, **10**, nr. 1, pp. 144–164, 1975.

[16] GOHBERG, I. Ţ., KRUPNIK, N. I., *O simvole singulearnh integralinh operatorovna slojnom conture*, Trud Tbilisscogo matematiciescogo instituta AN Gruz. SSR, **1**, pp. 46–59, 1973.

[17] PRESDORF, Z., *Necotore class singulearnh uravnenii*, M. Mir, 1979.

[18] ANTONEVICI, A. B., RÎVKIN V. B., *Ob uslovieah fredgolimovosti odnogo singulearnogo integralinogo operatora so sdvigom*, DAN BSSR, **19**, nr. 1, pp. 5–7.

[19] KARAPETEANŢ, N. C., SAMCO S. G., *Singulearne integraline operator so sdvigom Carlemana v sluciae cusocino-neprervnh coeffiţientov*, Izv. Vuzov., Matem., **2**, pp. 43–54, 1975.

[20] KARLOVICI, I. I., KRAVCIENCO, V. G., *O sistemah funcţionalinh i integrofuncţionalinh uravnenii s necarlemanovscim sdvigom*, DAN SSSR, **236**, nr. 5, pp. 1064–1067, 1977.

[21] KRUPNIK, N. I., NEAGA V. I., *O singulearnh operatorah so sdvigom v sluciae cusocinoleapunovscogo contura*, Soobşcenie AN Gruz. SSR, **76**, nr. 1, pp. 25–28.

[22] KRUPNIK, N. I., NEAGA V. I., *Singulearne integraline operator so sdvigom vdoli cusocino-leapunovscogo contura*, Izv. Vuzov., Matematica, nr. 6, pp. 60–72, 1975.

[23] KARAPETEANŢ, N. C., SAMCO, S.G., *Uravneniea s involiutivnmi operatorami i ih prilojeniea*, Izd-vo Rostovscogo un-ta, 1988.

[24] SOLDATOV, A. P., *Odnomerne singulearne operator i craeve zadaci teorii funcţii*, Moskva, Vssaia Şcola, 1991.

[25] NEAGA, V. I., *Singulearne integraline uravneniea so sdvigom vdoli neograniciennogo contura*, Izv. Vuzov., Matematica, nr. 5, pp. 35–42, 1982.

[26] NYAGA, V., *Singular integral operators. I. Case of an unlimited contour*, Buletinul AŞ a RM, nr. 1, 1999.

[27] BABENKO, C. I., O sopreajionnh funcţiah DAN SSSR, **62**, nr. 2, pp. 157–160, 1948.

[28] WIDOM, H., *Singular integral equation in L_p spaces*, Trans. Amer. Math. Soc., **97**, nr. 1, pp. 131–160, 1960.

[29] NEAGA, V. I., *O singulearnh integralinh operatorah v sluciae neogranicienogo contura*, Issled. po algebre, mat. analizu i ih prilojeniea Matem nauki, Chişinău, Ştiinţa, pp. 91–95, 1977.

[30] PICHORIDES, S. H., *On the best value of constants in the theoreme of M. Riesz, Zygmund and Kolmogorov*, Studia Math., **44**, nr. 2, pp. 165–179, 1972.

[31] ZIGMUND, A., *Trigonometricieschie read*, M., Mir, Vol. II, 1965.

[32] KRUPNIK, N. I., *Banahove algebr s simvolom i singulearne integraline operator*, Chişinău, Ştiinţa, 1984.

[33] NEAGA, V. I., *O factor-norme singularnh operatorov v sluciae slojnogo contura*, Izv. Vuzov., Matematica, **8**, pp. 74–79, 1978.

[34] SIMONENCO, I. B., *Craevaea zadacia Rimanna dlea n par funcţii s izmerimmi coeffiţientami i eë primenenie k issledovaniiu singulearnh integralov v prostranstvah L_p s vesami*, DAN SSSR, **141**, nr. 1, pp. 36–39, 1961.

[35] CLANCEY, K., GOHBERG, I., *Localization of singular integral operators. Integral equations and Operators Theory*, Birkhäuser Verlag, CH-4010 Basel (Switzerland), Vol. 3/1, 1980.

[36] NEAGA, V. I., *Matricine singulearne operator na slojnom conture*, Mat. Zametki AN SSSR, Vol. **30**, nr. 4, pp. 553–560, 1981.

[37] NEAGA, V. I., *O singulearnh integralinh uravnenieah v prostranstvah s vesom*, Izv. Vuzov., Matem., No. 2, pp. 73–75, 1979.

[38] NEAGA, V. I., *O singulearnh integralinh operatorah v prostrantvah s vesom Iddled. operaţii i programmirovaniea*, Matem. nauki Chişinău, Ştiinţa, pp. 94–102, 1982.

[39] NEAGA, V. I., *O singulearnh integralinh operatorah vdoli slojnogo contura*, Mat. Zametki AN SSSR, **21**, nr. 3, pp. 409–414, 1977.

[40] SIMONENCO, I. B., *Necotore obşcie vopros craevoi yadaci Rimanna*, Izv. AN SSSR, Ser. Matem., **32**, nr. 6, pp. 1138–1146, 1968.

[41] SIMONENCO, I. B., *Novi obşcii metod issledovaniea lineinh operatornh uravnenii tipa singulearnh integralinh uravnenii*, Izv. AN SSSR, Ser. Matem., **29**, nr. 3, pp. 567–586, 1965.

[42] KRUPNIK. N. I, *O factor-norme singulearnogo operatora*, matem. Issled, Chişinău, Ştiinţa, **10**, nr. 2, pp. 255–263, 1975.

[43] NEAGA, V. I., *Zavisimosti uslovia nëterovosti singulearnh operatorov so sdvigom ot nalicia uglovh tociec na conture. Sovremenne vopros pricladnoi matematiki i programmirovaniea*, Matem. nauki, Chişinău, Ştiinţa, pp. 99–102, 1979.

[44] NEAGA, V. I., *O singulearnom integralinom oepratore s sopreajeniem vdoli contura s uglovmi tocicami. Issledovaniea po differenţialinm uravnenieam*, Matem. issled, Chişinău, Ştiinţa, pp. 47–51, 1983.

[45] BOEARSKI, B. V., *Ob obobşcionnoi granicinoi zadaci Hilberta*, Soobscenie AN Gruz. SSR, **25**, nr. 4, pp. 385–390, 1960.

[46] MIHAILOV, L. G., *Novi class osobh integralinh uravnenii i ego prilojenie k differenţialinm uravnenieam s singulearnmi coeffiţientami*, Duşanbe, 1963.

[47] NYAGA, V. I., *The symbol of singular integral operators with conjugation the case of piecewise Lyapunov contour*, American Math. Society, **27**, no. 1, pp. 173–176, 1985.

[48] NYAGA, V. I., *Usloviea nioterovosti singulearnh integralinh operatorov s sopreajeniem v sluciae cusocino-leapunovscogo contura*, Issledovania po funcţ. analizu i differenţialinm uravnenieam, Chişinău, Ştiinţa, pp. 90–102, 1984.

[49] NYAGA, V. I., *Simvol singulearnh operatorov s sopreajeniem v sluciae cusocino-leapunovscogo contura*, DAN SSSR, **268**, nr. 4, pp. 806–808, 1984.

[50] NYAGA, V. I., *Vozmuşcenie singulearnh operatorov s cusocino-neprervnmi coeffiţientami. Issled, po funcţ. analizu i dif. uravnenieam*, Matem. nauki, Chişinău, Ştiinţa, pp. 64–68, 1978.

[51] NYAGA, V. I., *Ob odnom classe integralinh operatorov tipa singulearnh Nesamosopreajionne operator*, Matem. Issled, Chişinău, Ştiinţa, **42**, pp. 138–151, 1976.

[52] NYAGA, V. I., *Din Bat Tham. Ob odnom classe integralinh operatorov v prostranstvah s vesom*, Spectraline svoistva operatorov, Matem. Issled., Chişinău, Ştiinţa, 45, pp. 138–143, 1977.

[53] KRUPNIK, N. I., *Uslovie cvazinilipotentnosti integralinh operatorov. Vozmuşcenie operatorov localinogo tipa*, DAN SSSR, **234**, nr. 4, pp. 754–757, 1977.

[54] VASILESKII, N. L., *Ob odnom classe singulearnh integralinh operatorov s iadrom polearno-logarifmiciescogo tipa*, Izv. AN SSSR, Ser. Matem., **40**, nr. 1, pp. 133–151, 1976.

Received March 10, 1999