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ON THE EFFICIENCY OF THE APPROXIMATION OF INTEGRALS WITH HERMITE'S ALGORITHM

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Abstract. The numerical analyst is called upon to calculate an integral of continuous functions using numerical algorithms. This problem may be solved in many ways. An interesting approach is to consider the requested functions as a member of a linear space on which a probability measure is constructed and then to use established techniques of probability theory to determine the average cost of these algorithms. This work is concerned with finding the average cost for Hermite's rule. This work was suggested to me by S. Smale in the paper "on the efficiency of algorithms of analysis" (see [3]).

1. Suppose that on a Hilbert space H is given its Gaussian measure (see [2]). The "average" will refer to this measure. The following result is known:

PROPOSITION 1. *Let $L : H \to R$ be a bounded linear functional. Then the average satisfies:*

$$\operatorname*{Av}_{f\in H} |\, L\, x \,| = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|L\| \,.$$

(see [1,3]).

Here $<,>$ denotes the inner product in H. Choose $L^{\hat{}} \in H$ ($L^{\hat{}}$ the corresponding duals) so that $Lx = < L^{\hat{}}, x >$ for all $x \in H$. Then $\|L\| = \|L^{\hat{}}\|$ (Riesz' theorem).

2. Let us denote $C[0,1]$ the space of continuous functions on the interval [0,1] and $C^1[0,1]$ the space of class one on the interval [0,1].

Let $I : C[0,1] \to R$ denote the integral: $I\,(f) = \int\limits_0^1 f\,(s)\mathrm{d}s$.

For step size h, $h = 1/n$, $n \in N\backslash\{0\}$ there exists Hermite's rule:

$$S\,(h,f) = \left(\frac{h}{2}\right) \left[f\,(0) + f\,(1) + 2\sum_{i=1}^{n-1} f\,(ih) \right] + \left(\frac{h^2}{12}\right) [f'\,(0) - f'(1)]$$

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for $f \in C^1[0,1]$.

For step h the error in computing $I(f)$ with the Hermite's algorithm is given by:

$$R(h,f) = |I(f) - S(h,f)|.$$

For averaging these error functions, we need a probability on the functions space, then the Gaussian measure on Hilbert space is used.

The Hilbert space of functions we use is Sobolev's space $\mathcal{H}$.

Let f', f'' be the first and second derivatives of f, respectively. Then

$$\mathcal{H} = \left\{ f \in C^1[0,1] \mid f \text{ is defined almost everywhere and } \int_0^1 |f''(s)|\, ds < \infty \right\}$$

with inner product:

$$< f,g >= f(0)\, g(0) + f'(0)\, g'(0) + \int_0^1 f''(s)\, g''(s)\, ds.$$

Now it is possible to average the errors displayed above and therefore we obtain:

$$R(h) = \operatorname*{Av}_{f \in \mathcal{H}} R(h,f).$$

THEOREME 1. *For step size h, for $f \in \mathcal{H}$ we obtain* $R(h) = \dfrac{h^2}{6\sqrt{10\pi}}$.

3. We will slightly simplify the proof of the theorem, working with the subspace:

$$\mathcal{HO} = \{ f \in \mathcal{H} \mid f(0) = 0,\, f'(0) = 0 \}.$$

The inner product for $\mathcal{HO}$ is:

$$< f,g >= \int_0^1 f''(s)\, g''(s)\, ds.$$

Let $L : \mathcal{HO} \to R$, $L(f) = I(f) - S(h,f)$, for all $f \in \mathcal{HO}$.

Next, for $i = 0, ..., n$ we define linear functionals:

$$E(i,h) : \mathcal{HO} \to R \quad E(i,h)(f) = f(ih), \text{ for all } f \in \mathcal{HO} \text{ and}$$
$$D : \mathcal{HO} \to R \quad D(f) = f'(1), \text{ for all } f \in \mathcal{HO}$$

Then we have:

$$L = I + (h/2)\, E(n,h) - h \sum_{i=1}^{n} E(i,h) + \frac{h^2}{12} D.$$

Thus there exists $I\hat{}$, $L\hat{} \in \mathcal{HO}$ with

$$I(f) =< I\hat{}, f > \text{ and } L(f) =< L\hat{}, f >, \text{ for all } f \in \mathcal{HO}.$$

PROPOSITION 2. *It is true that:*

(i) $(I\hat{})''(s) = \left(\dfrac{(s-1)^2}{2} \right)$, $s \in [0,1]$.

(ii) $(E(i,h)\hat{})''(s) = \begin{cases} ih - s & \text{for } s \le ih \\ 0 & \text{for } s \ge ih \end{cases}$, $s \in [0,1]$, $i = 0, ..., n$.

(iii) $(D\hat{})''(s) = 1$, $s \in [0,1]$.

Proof. The proof is easy calculus. Let $f \in \mathcal{HO}$. For (i) it amounts to checking.

$$< I\hat{}, f > = \int_0^1 (I\hat{})''(s)\, f''(s)\, ds = \int_0^1 \left((s-1)^2/2 \right) f''(s)\, ds = $$
$$= \int_0^1 f(s)\, ds = I(f).$$

For (ii) we have:

$$< E(i,h)\hat{}, f > = \int_0^1 (E(i,h)\hat{})''(s)\, f''(s)\, ds = \int_0^{ih} f''(s)\, ds =$$
$$= f(ih) = E(i,h)(f).$$

For (iii) we have:

$$< D\hat{}, f >= \int_0^1 (D\hat{})''(s)\, f''(s)\, ds = \int_0^1 f''(s)\, ds = f(1) = D(f).$$

□

PROPOSITION 3. *It is true that:*

$$\| I - S(h,f) \|^2 = \frac{h^4}{720}.$$

Proof. The proof is calculus for $|I\hat{} - S(h,f)\hat{}|$ because $|I - S(h,f)| = |I\hat{} - S(h,f)\hat{}|$.

Then we have:

$$\| I\hat{} - S(h,f)\hat{} \|^2 = \int_0^1 ((L\hat{})(s))^2 ds = \int_0^1 \left[\frac{(s-1)^2}{2} - h \sum_{i=1}^{n} (E(i,h)\hat{})''(s) + \right.$$
$$\left. + \frac{h}{2} (E(n,h)\hat{})''(s) + \frac{h^2}{12} (D\hat{})''(s) \right] ds.$$

For $s \in [(j-1)s, js]$ and $j = 1, ..., n$ we have:

$$\sum_{i=1}^{n} (E(i,h)\hat{\ })''(s) = \sum_{i=j}^{n} (ih - s) = \frac{h}{2}[n(n+1) - j(j-1)] - s(n-j+1).$$

We conclude that:

$$\|I\hat{\ } - S(h,f)\hat{\ }\|^2 = \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \left\{ \frac{(s-1)^2}{2} + \frac{h^2}{12} + h\frac{(1-s)}{2} - h\left[\frac{n+1}{2} - h\frac{j(j-1)}{2} - s(n-j+1)\right]\right\}^2 ds,$$

$$\|I\hat{\ } - S(h,f)\hat{\ }\|^2 = \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \left[a(j)s^2 + b(j)s + c(j)\right]^2 ds$$

with

$$a(j) = \frac{1}{2},\ b(j) = h\frac{1-2j}{2},\ c(j) = \frac{h^2}{12} + h^2\frac{j(j-1)}{2}.$$

Calculating we obtain:

$$\|I\hat{\ } - S(h,f)\hat{\ }\|^2 = \sum_{j=1}^{n} \left\{ a(j)\frac{s^5}{5} + a(j)b(j)\frac{s^4}{2} + \left[b(j)^2 + 2a(j)c(j)\right]\frac{s^3}{3} + b(j)c(j)s^2 + c(j)^2 s \right\}\Big|_{(j-1)h}^{jh}$$

$$\|I\hat{\ } - S(h,f)\hat{\ }\|^2 = \sum_{j=1}^{n} \frac{h^5}{720} = \frac{h^4}{720},$$

which proves the proposition. □

4. *Proof of the Theorem*

In the *Proposition 1* for $H = \mathcal{H}$ and $L = I - S(h,f)$ we obtain:

$$R(h) = \operatorname*{Av}_{f \in \mathcal{H}} R(h,f),$$

$$R(h) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|I\hat{\ } - S(h,f)\hat{\ }\|,$$

$$R(h) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{h^2}{12\sqrt{5}} = \frac{h^2}{6\sqrt{10\pi}},$$

finishing the proof of the theorem.

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