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SOME SEQUENCES SUPPLIED BY INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In order to prove the convergence of Ishikawa and Mann iterations, the convergence of one type of sequences is needed. Our purpose, in this note, is to give a new proof of the convergence for one of them. We also give generalizations for the sequences.

Key words: nonnegative real sequences.

1. INTRODUCTION

The convergence of Mann, Ishikawa iterations are studied in the papers [3], [4], [5]. This iterations are approximating the fixed point of strictly psedo-contractive mapping. One role in the convergence of them is given by some real nonnegative sequences, which are verifying one type of inequalities.

In Proposition 2.1, we give another proof to Lemma 2.1 from [4]. The proof from [4] of Lemma 2.1 is similar to that of Lemma 1.2 from [5]. Lemma 2.1 is used to prove the convergence of the Ishikawa iteration in [4].

In Proposition 2.2 we will study the convergence of the real, nonnegative sequence (a_n), given by the following recurrence:

$$a_{n+1} \leq (1-w)a_n + \alpha_n M,$$

where $w \in (0,1]$, $M > 0$ are fixed numbers, and $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\alpha_n \to 0$.

Proposition 2.3, (see [3]), is a generalization of Proposition 2.2, but the sequence isn't necessary convergent to zero. The sequence (a_n) is given by the following recurrence:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n M,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, and $M > 0$ is fixed.

In another context, Proposition 2.4 generalizes Proposition 2.2. The sequence (a_n) is given by the recurrence:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n,$$

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where $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (c_n) is a nonnegative real sequence, increasing and bounded.

Proposition 2.5 is another generalization of Proposition 2.2. In this context the sequence (a_n) is given by the recurrence:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n,$$

where $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (c_n) is a nonnegative real sequence and $\sum_{n=1}^{\infty} \alpha_n c_n = l$. The sequence (a_n) isn't necessary convergent to zero.

2. MAIN RESULTS

Below we will give a new proof to Lemma 2.1 from [4].

PROPOSITION 2.1. [4]. *Let (a_n), (b_n) and (c_n) be three nonnegative and real sequences which verify the following inequality*

$$a_{n+1} \leq \sqrt{(1-w)a_n^2 + b_n t_n + c_n},$$

where $t_n \in [0,1]$, $\forall n \in \mathbb{N}$, and $w \in (0,1]$ is fixed, $b_n \to 0$, $c_n \to 0$. Then $a_n \to 0$, as $n \to \infty$.

Proof. From our assumptions there exist the following numbers:

$$t = \sup\{t_n : n \geq 1\},$$
$$s_1 = \sup\left\{\frac{2b_n}{w} t : n \geq 1\right\}$$
$$s_2 = \sup\left\{\frac{4c_n}{w} : n \geq 1\right\}$$
$$m^2 = \max\{a_1^2, s_1, s_2^2\}.$$

We will prove that $a_n \leq m$, $\forall n \in \mathbb{N}$. We can see that $a_1 \leq m$. We suppose that $a_n \leq m$ is true, and we prove that $a_{n+1} \leq m$. We have

$$a_{n+1} \leq \sqrt{(1-w)m^2 + \frac{ws_1}{2t}t + \frac{ws_2}{4}} \leq \sqrt{(1-w)m^2 + \frac{w}{2}m^2} + \frac{w}{4}m =$$
$$= \left(\sqrt{1-\frac{w}{2}}\right) m + \frac{w}{4}m \leq m.$$

The last inequality is true because

$$\sqrt{1-\frac{w}{2}} + \frac{w}{4} \leq 1.$$

Hence the sequence is bounded, so the limit exists $a = \limsup_{n\to\infty} a_n$, and $a < +\infty$. Hence $a \leq \sqrt{1-w}a$. Thus $a = 0$. □

The proof of the next result is similar to the proof of Proposition 2.1.

PROPOSITION 2.2. *Let (a_n) be a nonnegative and real sequence which verifies the following inequality*

$$a_{n+1} \leq (1-w)a_n + \alpha_n M,$$

where $w \in (0,1]$, $M > 0$ are fixed numbers, and $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\alpha_n \to 0$. Then $a_n \to 0$ as $n \to \infty$.

Proof. From our assumptions there exist the following numbers:

$$s := \sup_{n\in\mathbb{N}}\{\frac{\alpha_n M}{w}\},$$
$$m := \max\{a_1, s\}.$$

We will prove that $a_n \leq m$, $\forall n \in \mathbb{N}$. We can see that $a_1 \leq m$. We suppose that $a_n \leq m$ is true, and we prove that $a_{n+1} \leq m$. We have

$$a_{n+1} \leq (1-w)m + s\,w \leq (1-w)m + m\,w = m.$$

The sequence (a_n) is bounded. Then there exists $a = \limsup_{n\to\infty} a_n$ and $a < +\infty$. Hence $a \leq (1-w)a + 0$. Thus $a = 0$. □

The next result is Lemma 1 from [3]. Here, instead of w, we have the sequence (α_n).

PROPOSITION 2.3. [3]. *Let (a_n) be a nonnegative and real sequence which verifies the following inequality*

(1) $$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n M,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$ and $M > 0$ is fixed. Then

$$0 \leq \lim_{n\to\infty} \sup a_n \leq M.$$

Proposition 2.3 is used to prove the convergence of the Mann iteration in [3].

Remark 2.1. Under the same assumptions as in Proposition 2.3, even if $\alpha_n \to 0$, we can have $\lim_{n\to\infty} a_n \neq 0$.

For $M = 1$, $\alpha_n = \dfrac{1}{n+1}$, $a_n = \dfrac{n-1}{n}$ in Proposition 2.3, we see that a_{n+1} verifies the recurrence relation and the inequality (1).

$$a_{n+1} = \left(1 - \frac{1}{n+1}\right)\frac{n-1}{n} + \frac{1}{n+1} = \frac{n}{n+1}.$$

The sequence (a_n) does not converge to zero, $\lim_{n\to\infty} a_n = \lim\sup_{n\to\infty} a_n = 1$.

Corollary 2.1. *Let (a_n) be a nonnegative and real sequence which verifies the following inequality*

$$a_{n+1} \leq (1-\alpha)a_n + \alpha\alpha_n,$$

where $\alpha \in (0,1]$ is a fixed number and $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\alpha_n \to 0$. Then $a_n \to 0$ as $n \to \infty$.

Proof. We put $\alpha := M = w$ in Proposition 2.2. □

Lemma 2.1. *Let $(\beta_n)_n$ be a nonnegative and real sequence such that $\beta_n \in (0,1]$, $\forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n = \infty$, then $\prod_{n=1}^{\infty}(1-\beta_n) = 0$.*

Proof. For $n = 1$ we get $1 - \beta_1 \leq \frac{1}{1+\beta_1}$. Suppose the following is true

$$(1-\beta_1)\dots(1-\beta_n) \leq \frac{1}{1+\beta_1+\dots+\beta_n}.$$

For $n+1$ we have

$$\begin{aligned}(1-\beta_1)\dots(1-\beta_n)(1-\beta_{n+1}) &\leq \frac{1-\beta_{n+1}}{1+\beta_1+\dots+\beta_n} = \frac{1}{1+\beta_1+\dots+\beta_n} - \\ &- \frac{\beta_{n+1}}{1+\beta_1+\dots+\beta_n} \leq \\ &\leq \frac{1}{1+\beta_1+\dots+\beta_n+\beta_{n+1}}.\end{aligned}$$

Obviously, the last inequality holds. Indeed, we have

$$\begin{aligned}\frac{1}{1+\beta_1+\dots+\beta_n} &\leq \frac{\beta_{n+1}}{1+\beta_1+\dots+\beta_n} + \frac{1}{1+\beta_1+\dots+\beta_n+\beta_{n+1}} \Leftrightarrow \\ &\Leftrightarrow 1+\beta_1+\dots+\beta_n+\beta_{n+1} \leq \\ &\leq \beta_{n+1}(1+\beta_1+\dots+\beta_n+\beta_{n+1}) + (1+\beta_1+\dots+\beta_n) \Leftrightarrow \\ &0 \leq \beta_{n+1}(\beta_1+\dots+\beta_n+\beta_{n+1});\end{aligned}$$

the latter is, of course, true, $\forall \beta_i \in (0,1)$, and $i \in \{1,\dots n, n+1\}$. Thus, one obtains

$$\prod_{k=1}^{n}(1-\beta_k) \leq \frac{1}{1+\sum_{k=1}^{n}\beta_k} \leq \frac{1}{\sum_{k=1}^{n}\beta_k}, \ \forall n \in \mathbb{N}.$$

□

Proposition 2.4. *Let $(a_n)_n$ be a nonnegative and real sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n,$$

where $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and (c_n) is a nonnegative real sequence, increasing and bounded. Then

$$0 \leq \limsup_{n\to\infty} a_n \leq \lim_{n\to\infty} c_n. \tag{2}$$

Proof. By mathematical induction, we shall have

$$a_{n+1} \leq (1-\alpha_n)(1-\alpha_{n-1})\dots(1-\alpha_1)a_1 + c_n. \tag{3}$$

Indeed, for $n = 1$, we get $a_2 \leq (1-\alpha_1)a_1 + \alpha_1 c_1 \leq (1-\alpha_1)a_1 + c_1$. Suppose the inequality holds for n. Then

$$\begin{aligned}a_{n+2} &\leq (1-\alpha_{n+1})a_{n+1} + \alpha_{n+1}c_{n+1} \leq \\ &\leq (1-\alpha_{n+1})[(1-\alpha_n)(1-\alpha_{n-1})\dots(1-\alpha_1)a_1 + c_n] + \alpha_{n+1}c_{n+1} \leq \\ &\leq (1-\alpha_{n+1})(1-\alpha_n)\dots(1-\alpha_1)a_1 + (1-\alpha_{n+1})c_n + \alpha_{n+1}c_{n+1} \leq \\ &\leq (1-\alpha_{n+1})(1-\alpha_n)\dots(1-\alpha_1)a_1 + c_{n+1}.\end{aligned}$$

As (c_n) is increasing, we have $(1-\alpha_{n+1})c_n \leq (1-\alpha_{n+1})c_{n+1}$. From Lemma 2.1, we have: $\sum_{n=1}^{\infty}\alpha_n = \infty \Rightarrow \prod_{n=1}^{\infty}(1-\alpha_{n+1}) = 0$. By Weierstrass's theorem, there exists $\lim_{n\to\infty} c_n$. Taking $n \to \infty$ from (3), we arrive to conclusion (2). □

Proposition 2.5. *Let $(a_n)_n$ be a nonnegative and real sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n,$$

where $\alpha_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and (c_n) is a nonnegative real sequence and $\sum_{n=1}^{\infty}\alpha_n c_n = l$. Then

$$0 \leq \limsup_{n\to\infty} a_n \leq l. \tag{4}$$

Proof. By mathematical induction, we shall have

$$a_{n+1} \leq \prod_{k=1}^{n}(1-\alpha_k)a_1 + \sum_{k=1}^{n}\alpha_k c_k. \tag{5}$$

Indeed, for $n = 1$, we get $a_2 \leq (1-\alpha_1)a_1 + \alpha_1 c_1$. Suppose the inequality holds for n. Then

$$\begin{aligned} a_{n+2} &\leq (1-\alpha_{n+1})a_{n+1} + \alpha_{n+1}c_{n+1} \leq \\ &\leq (1-\alpha_{n+1})\left[\prod_{k=1}^{n}(1-\alpha_k)a_1 + \sum_{k=1}^{n}\alpha_k c_k\right] + \alpha_{n+1}c_{n+1} \leq \\ &\leq \prod_{k=1}^{n+1}(1-\alpha_k)a_1 + (1-\alpha_{n+1})\sum_{k=1}^{n}\alpha_k c_k + \alpha_{n+1}c_{n+1} \leq \\ &\leq \prod_{k=1}^{n+1}(1-\alpha_k)a_1 + \sum_{k=1}^{n+1}\alpha_k c_k. \end{aligned}$$

As $(1-\alpha_{n+1}) \in (0,1)$, we have $(1-\alpha_{n+1})\sum_{k=1}^{n}\alpha_k c_k \leq \sum_{k=1}^{n}\alpha_k c_k$. From Lemma 2.1, we have: $\sum_{n=1}^{\infty}\alpha_n = \infty \Rightarrow \prod_{n=1}^{\infty}(1-\alpha_n) = 0$. Taking $n \to \infty$ from (5), we arrive to conclusion (4). □

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