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AN APPLICATION OF THE FIXED POINT THEOREM OF BOHNENBLUST-KARLIN TO THE DARBOUX PROBLEM FOR A MULTIVALUED INCLUSION

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Abstract. In this paper we consider the Darboux problem

$$(0.1)\qquad \frac{\partial^2 z}{\partial x \partial y} \in F(x,y,z),\ \ z(x,0)=\sigma(x),\ \ z(0,y)=\tau(y),\ \ \sigma(0)=\tau(0),$$

where $F : D \times E^n \to 2^{E^n}$ is a multifunction, $D=[0,a]\times[0,b]$, E^n is the Euclidean n-space and $\sigma \in C^1([0,a],E^n)$, $\tau \in C^1([0,b],E^n)$. It is defined the notion of *classical solution* for the problem (0.1) and it is proved an existence theorem for such a solution using the fixed point theorem of Bohnenblust-Karlin for multivalued applications. The paper is an extension of [4]. The Darboux problem with F also depending of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\alpha_i(x,y)$, $\beta_i(x,y)$, $i=\overline{1,\nu}$, and λ – a parameter, with solutions defined in various ways as absolutely continuous functions, with classical or generalized solutions was studied in [3], [8], [9], [10], [12], [14]–[34].

1. PRELIMINARIES

The Euclidean distance between two points z_1, $z_2 \in E^n$ will be denoted [4]

$$\rho(z_1,z_2)=|z_1-z_2|.$$

DEFINITION 1.1. [4], [9], [10]. *If $A \subset E^n$, the distance from $z \in E^n$ to the set A is*

$$\rho(z,A)=\inf\{\rho(z,a) : a \in A\}.$$

DEFINITION 1.2. [6]. *A neighbourhood of the set $A \subset E^n$ is*

$$N_\varepsilon(A)=\{z \in E^n : \rho(z,a)<\varepsilon,\ a \in A,\ \varepsilon>0\}.$$

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DEFINITION 1.3. [4], [6], [9], [10], [13], [15]. *If A and B are compact subsets of a metric space X, the Hausdorff-Pompeiu metric h is defined thus: h is the smallest positive real number d such that A is contained in a d-neighbourhood of B and B in a d-neighbourhood of A:*

$$h(A,B) = \inf\{d \in \mathbb{R}_+ : A \subseteq N_d(B) \quad \text{and} \quad B \subseteq N_d(A)\}.$$

THEOREM 1.1. [4], [11]. *The set $\Omega^n = \text{comp}E^n$ of all nonempty, compact subsets of E^n, with the topology induced by the Hausdorff-Pompeiu metric, is a complete metric space, (Ω^n, h).*

DEFINITION 1.4. [6], [9], [10], [13]. *Let X, Y be two nonempty sets. A multifunction $F : X \to 2^Y$ is a function from X into the family of all nonempty subsets of Y.*

DEFINITION 1.5. [4], [6]. *Let $D = [0,a] \times [0,b] \subset \mathbb{R}^2$. A multifunction $G : D \to \Omega^n$ is* measurable *(in the sense of Lebesque) if for every closed subset $\Delta \subset E^n$, the set*

$$\Delta^- = \{(x,y) \in D : G(x,y) \cap \Delta \neq \varnothing\}$$

is Lebesgue-measurable.

DEFINITION 1.6. [6], [9], [10]. *If T is a topological space and Y a metric space, the multifunction $F : T \to 2^Y$ is upper semicontinuous (lower semicontinuous) if for every closed (open) suset $B \subseteq Y$, the set*

$$\{t \in T : F(t) \cap B \neq \varnothing\}$$

is closed (open) in T.

DEFINITION 1.7. [7]. *The multifunction $F : T \to 2^Y$ is continuous if it is upper and lower semicontinuous.*

DEFINITION 1.8. [1], [4], [18]. *Let be $G : D \to \Omega^n$. The Aumann integral of the set-valued function G is defined by*

$$\int_D\int G(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= \left\{\int_D\int g(x,y)\mathrm{d}x\mathrm{d}y : \ g \text{ measurable}, \ g(x,y) \in G(x,y), \ (x,y) \in D\right\}.$$

DEFINITION 1.9. [4]. *For a mapping $G : D \to \Omega^n$, it is defined* co$G$ *: $D \to \Omega^n$, to be the map from D into Ω^n taking values* co$G(x,y)$ *as the closed convexed hull of $G(x,y)$, $(x,y) \in D$.*

We shall use the notation $C(D, E^n)$ for the space of continuous n-vector valued functions, defined on D, with the uniform topology, i.e. the topology induced by $\|g\| = \max\{|g(x,y)| : (x,y) \in D\}$, where $|g(x,y)|$ is the Euclidean norm of $g(x,y) \in E^n$.

Lemma 1.4 of [4] can be easily extended for functions in two variables as follows:

LEMMA 1.1. *Let $G : D \to \Omega^n$ be measurable, with values in a ball $B_\rho(\theta)$ centered at θ-the origin of E^n, of radius ρ. Let us consider the sets*

$$Y = \{g \in L_\infty(D, E^n) : g(\xi,\eta) \in \text{co}\, G(\xi,\eta), \ (\xi,\eta) \in D\}$$

and

$$\mathcal{B} = \{h \in C(D, E^n) : h(x,y) = \alpha(x,y) + \int_0^x\int_0^y g(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta, \ g \in Y\}.$$

Then $\mathcal{B}$ is a compact, convex subset of $C(D,E^n)$.

Then $\mathcal{B}$ is a compact, convex subset of $C(D,E^n)$.

THEOREM 1.2. (Bohnenblust-Karlin) [4]. *Let S be a compact convex subset of a Banach space X and $A : S \to CCl(S)$ [15] a continuous (in the Hausdorff topology) mapping from S into the space of nonempty closed convex subsets of S. Then A has a fixed point, i.e. there exists a $\varphi \in S$ such that $\varphi \in A(\varphi)$. See also [2], [7], [13].*

If $I \subset \mathbb{R}$ is a compact interval and X is a Banach space, we denote by $C^1(I,X)$ [10] the space of all continuously differentiable functions from I into X, endowed with the norm

$$\|\varphi\|_{C^1(I,X)} = \max_{t\in I}\|\varphi(t)\| + \max_{t\in I}\|\varphi' I(t)\|.$$

2. RESULTS

Let the following hypothesese satisfied:

(H_1) F : D × $E^n \to \Omega^n$ is a multifunction with its values $F(x,y,z)$ for $(x,y) \in D$, $z \in E^n$ as compact convex sets in E^n contained in the ball of radius L.

(H_2) The functions $C^1([0,a],E^n)$, $\tau \in C^1([0,b],E^n)$ satisfy the condition $\sigma(0) = \tau(0)$.

(H_3) The function $\alpha : D \to E^n$, defined by

$$\alpha(x,y) = \sigma(x) + \tau(y) - \sigma(0), \ \ (x,y) \in D, \tag{2.1}$$

is bounded. There exists $\alpha_0 \in \mathbb{R}_+$ such that

$$|\alpha(x,y)| \le \alpha_0, \ \ (x,y) \in D. \tag{2.2}$$

DEFINITION 2.1. [18], [19]. *The Darboux problem for the hyperbolic inclusion*

$$\frac{\partial^2 z}{\partial x \partial y} \in F(x,y,z), \ \ (x,y) \in D, \ z \in E^n, \tag{2.3}$$

consists in the determination of a solution for (2.3) satisfying conditions

$$(2.4)\quad \begin{cases} z(x,0)=\sigma(x), & 0\le x\le a \\ z(0,y)=\tau(y), & 0\le y\le b. \end{cases}$$

DEFINITION 2.2. [5], [10], [34]. *A function $z : D \to E^n$ is said to be a classical solution of (2.3)+(2.4) if $z, \dfrac{\partial^2 z}{\partial x\partial y}$ are in $C(D,E^n)$ and for every $(x,y)\in D$, z satisfies (2.3), i.e. $\dfrac{\partial^2 z(x,y)}{\partial x\partial y}\in F(x,y,z(x,y))$, and also (2.4).*

THEOREM 2.1. *Let the hypotheses* (H_1), (H_2), (H_3) *be satisfied. Then the Darboux problem (2.3)+(2.4) has a classical solution.*

Proof. Let $C_L(D,E^n)$ be the compact subset of $C(D,E^n)$ consisting of functions with Lipschitz constant L and of norm less than or equal to α_0+Lab.

$$C_L(D,E^n)=\{f\in C(D,E^n): |f(x,y)-f(x',y')|\le L(|x-x'|+|y-y'|,(x,y),(x',y')\in D, |f|\le \alpha_0+Lab\}.$$

Let z be in $C(D,E^n)$ and $\dfrac{\partial^2 z}{\partial x\partial y}$ in $C(D,E^n)$. Integrating $\dfrac{\partial^2 z}{\partial x\partial y}$ over the domain

$$D_{xy}=\{(\xi,\eta): 0\le\xi\le x,\ 0\le\eta\le y\}\subseteq D,\ (x,y)\in D,$$

and using (2.2), (2.4) one gets

$$(2.5)\quad z(x,y)=\sigma(x)+\tau(y)-\sigma(0)+\int_0^x\int_0^y \frac{\partial^2 z}{\partial\xi\partial\eta}\mathrm{d}\xi\mathrm{d}\eta = \alpha(x,y)+\int_0^x\int_0^y \frac{\partial^2 z}{\partial\xi\partial\eta}\mathrm{d}\xi\mathrm{d}\eta,\ (x,y)\in D.$$

For $Z\in C_L(D,E^n)$ let us take

$$Y(z)=\{f\in L_\infty(D,E^n): f(\xi,\eta)\in F(\xi,\eta,z(\xi,\eta)),(\xi,\eta)\in D\}$$

and

$$A(z)=\{h\in C(D,E^n): h(x,y)=\alpha(x,y)+\int_0^x\int_0^y f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,\ f\in Y(z)\}.$$

Since F has convex values, Lemma 1.1 shows that $A(z)$ is a compact convex subset of $C_L(D,E^n)$. Also, A considered as a map

$$A: C_L(D,E^n)\to \text{comp}\, C_L(D,E^n)$$

of $C_L(D,E^n)$ into the nonempty compact subsets of $C_L(D,E^n)$, is continuous in the Hausdorff topology. By the Bohnenblust-Karlin fixed point theorem, A

has a fixed point φ and φ is therefore a solution of the problem (2.3)+(2.4). Indeed

$$Y(\varphi)=\{f\in L_\infty(D,E^n): f(\xi,\eta)\in F(\xi,\eta,\varphi(\xi,\eta)),(\xi,\eta)\in D\}$$

and

$$A(\varphi)=\{h\in C(D,E^n): h(x,y)=\alpha(x,y)+\int_0^x\int_0^y f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,\ f\in Y(\varphi)\}.$$

From $A(\varphi)=\varphi$ it follows

$$\varphi(x,y)=h(x,y)=\alpha(x,y)+\int_0^x\int_0^y f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,$$

hence

$$\frac{\partial^2\varphi}{\partial x\partial y}(x,y)=f(x,y)\in F(x,y,\varphi(x,y)),\quad for (x,y)\in D.$$

i.e. (2.3) is satisfied, and (2.4) obviously holds. One has

$$\begin{cases} \dfrac{\partial\varphi}{\partial x}(x,0)=\sigma(x), & 0\le x\le a \\ \dfrac{\partial\varphi}{\partial y}(0,y)=\tau(y), & 0\le y\le b. \end{cases}$$

The theorem 2.1 is thus proved. □

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