REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 29 (2000) no. 2, pp. 221–229
ictp.acad.ro/jnaat

MEYER-KÖNIG AND ZELLER OPERATORS BASED ON THE q-INTEGERS

TIBERIU TRIF

Abstract. By means of the q-integers as well as of the Gaussian binomial coefficients we introduce a generalization of the Meyer-König and Zeller operators. For a fixed number $q \in \,]0,1]$, the sequence of the generalized Meyer-König and Zeller operators is denoted by $(M_{n,q})_{n\geq 1}$. Both a theorem on convergence and a Popoviciu type theorem on the rate of convergence are proved. It is shown that if f is increasing, then $M_{n,q}f$ is also increasing, while if f is convex, then $M_{n,q}f$ is also convex and $M_{n,q}f \geq f$, generalizing known results when $q = 1$. Likewise, it is shown that if f is convex, then the sequence $(M_{n,q}f)_{n\geq 1}$ is non-increasing as in the case of the classical M_n-operators.

1. INTRODUCTION

Let q be a number belonging to the interval $]0,1]$ (throughout the paper q will always have this meaning). For each non-negative integer k, the q-integer $[k]$ is defined by

$$[k] = \begin{cases} \frac{1-q^k}{1-q}, & \text{if } q \neq 1 \\ k, & \text{if } q = 1, \end{cases}$$

while the q-factorial $[k]!$ is defined by

$$[k]! = \begin{cases} [k]\cdot[k-1]\cdots[1], & \text{if } k \geq 1 \\ 1, & \text{if } k = 0. \end{cases}$$

By means of the q-factorial, the Gaussian binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}$$

for all integers $n \geq k \geq 0$.

1991 AMS Subject Classification: 41A36, 41A25.

It is easily seen that the Gaussian binomial coefficients satisfy the recurrence relations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix}$$

for all integers $n \geq k \geq 1$.

Based on the q-integers and on the Gaussian binomial coefficients G.M. Phillips [10] proposed the following generalization of the classical Bernstein operators: for each positive integer n let $B_{n,q} : C[0,1] \to C[0,1]$ be the operator defined by

$$B_{n,q} f(x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{j=0}^{n-k-1} (1 - q^j x),$$

where an empty product denotes 1. For $q = 1$, $B_{n,1}$ reduces to the classical Bernstein operator $B_n : C[0,1] \to C[0,1]$

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Approximation properties of these generalized Bernstein operators, which are quite similar with those of the classical Bernstein operators, were investigated in [5], [9], and [10].

Starting from the power series expansion

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k, \qquad 0 \leq x < 1, \tag{1.1}$$

W. Meyer-König and K. Zeller [8] introduced the sequence $(M_n)_{n\geq 1}$ of linear positive operators $M_n : C[0,1] \to C[0,1]$ defined by

$$M_n f(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k, \qquad 0 \leq x < 1,$$
$$M_n f(1) = f(1).$$

For approximation properties of the M_n-operators the reader is referred to [3] and [7].

The q-generalization of (1.1) is the following power series expansion:

$$\frac{1}{\prod_{j=0}^{n} (1 - q^j x)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k, \qquad 0 \leq x < 1. \tag{1.2}$$

Simple proofs of (1.2) can be found in [1] and [2]. Starting from (1.2) we introduce the sequence $(M_{n,q})_{n\geq 1}$ of linear positive operators $M_{n,q} : C[0,1] \to C[0,1]$ defined by

$$M_{n,q}f(x) = P_{n,q}(x) \sum_{k=0}^{\infty} f\left(\tfrac{[k]}{[n+k]}\right) \left[{n+k \atop k}\right] x^k, \ 0 \leq x < 1, \tag{1.3}$$

$$M_{n,q}f(1) = f(1), \tag{1.4}$$

where

$$P_{n,q}(x) = \prod_{j=0}^{n} \left(1 - q^j x\right).$$

It is easy to check (see [8]) that

$$\lim_{x \nearrow 1} M_{n,q}f(x) = f(1)$$

for all $f \in C[0,1]$, so $M_{n,q}$ is well-defined for each positive integer n. In fact $M_{n,q}f$ can be defined by (1.3) for each bounded function $f : [0,1] \to \mathbb{R}$. If in addition there exists $\lim_{x \nearrow 1} f(x) = l$, then $\lim_{x \nearrow 1} M_{n,q}f(x) = l$. In this case (1.4) must be replaced by

$$M_{n,q}f(1) = \lim_{x \nearrow 1} f(x). \tag{1.5}$$

In analogy with the terminology used in [10], in what follows we will call the $M_{n,q}$-operators generalized Meyer-König and Zeller operators.

It is the main purpose of this paper to investigate the approximation properties of the above introduced operators.

2. THE ORDER OF APPROXIMATION BY GENERALIZED MEYER-KÖNIG AND ZELLER OPERATORS

For each nonnegative integer i let e_i denote the monomial $e_i(x) = x^i$.

LEMMA 2.1. *For all integers $n \geq 3$ and all $x \in [0,1]$ the following relations are valid:*

$$M_{n,q}e_0(x) = 1, \tag{2.1}$$

$$M_{n,q}e_1(x) = x, \tag{2.2}$$

$$M_{n,q}e_2(x) = x^2 + \tfrac{x(1-x)(1-q^n x)}{[n-1]} - R_{n,q}(x), \tag{2.3}$$

where

$$0 \le R_{n,q}(x) \le \tfrac{[2]q^{n-1}}{[n-1][n-2]} x(1-x)(1-qx)(1-q^n x). \tag{2.4}$$

Proof. Relation (2.1) is just (1.2), while relation (2.2) follows immediately from (1.2) taking into account that

$$\tfrac{[k]}{[n+k]} \left[{n+k \atop k}\right] = \left[{n+k-1 \atop k-1}\right] \tag{2.5}$$

for all positive integers n and k.

To prove (2.3)–(2.4) we fix an integer $n \ge 3$ as well as a number $x \in [0,1[$ ((2.3)–(2.4) are trivially true for $x = 1$). Taking into account (2.5) we get

$$\begin{aligned} M_{n,q}e_2(x) &= P_{n,q}(x) \sum_{k=1}^{\infty} \tfrac{[k]}{[n+k]} \left[{n+k-1 \atop k-1}\right] x^k \\ &= xP_{n,q}(x) \sum_{k=0}^{\infty} \tfrac{[k+1]}{[n+k+1]} \left[{n+k \atop k}\right] x^k. \end{aligned}$$

On the other hand, (1.2) yields

$$e_2(x) = xP_{n,q}(x) \sum_{k=1}^{\infty} \left[{n+k-1 \atop k-1}\right] x^k.$$

Consequently

$$M_{n,q}e_2(x) - x^2 = \tfrac{xP_{n,q}(x)}{[n+1]} + xP_{n,q}(x) \sum_{k=1}^{\infty} \left(\tfrac{[k+1]}{[n+k+1]} \left[{n+k \atop k}\right] - \left[{n+k-1 \atop k-1}\right] \right) x^k.$$

By a simple computation

$$\tfrac{[k+1]}{[n+k+1]} \left[{n+k \atop k}\right] - \left[{n+k-1 \atop k-1}\right] = \tfrac{q^k}{[n+k+1]} \left[{n+k-1 \atop k}\right],$$

so

$$\begin{aligned} M_{n,q}e_2(x) &= x^2 + xP_{n,q}(x) \sum_{k=0}^{\infty} \tfrac{q^k}{[n+k+1]} \left[{n+k-1 \atop k}\right] x^k \\ &= x^2 + \tfrac{xP_{n,q}(x)}{[n-1]} \sum_{k=0}^{\infty} q^k \tfrac{[n+k-1]}{[n+k+1]} \left[{n+k-2 \atop k}\right] x^k. \end{aligned}$$

Since

$$\tfrac{[n+k-1]}{[n+k+1]} = 1 - \tfrac{[2]q^{n+k-1}}{[n+k+1]},$$

we get

$$M_{n,q}e_2(x) = x^2 + \frac{xP_{n,q}(x)}{[n-1]} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} (qx)^k$$
$$- \frac{[2]q^{n-1}xP_{n,q}(x)}{[n-1]} \sum_{k=0}^{\infty} \frac{1}{[n+k+1]} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} \left(q^2x\right)^k.$$

Relation (1.2) ensures that

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} (qx)^k = \frac{1}{\prod\limits_{j=0}^{n-2} \left(1 - q^j \cdot qx\right)} = \frac{(1-x)(1-q^nx)}{P_{n,q}(x)},$$

hence

$$M_{n,q}e_2(x) = x^2 + \frac{x(1-x)(1-q^nx)}{[n-1]} - R_{n,q}(x),$$

where

$$\begin{aligned} 0 \le R_{n,q}(x) &= \frac{[2]q^{n-1}xP_{n,q}(x)}{[n-1]} \sum_{k=0}^{\infty} \frac{1}{[n+k+1]} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} \left(q^2x\right)^k \\ &\le \frac{[2]q^{n-1}xP_{n,q}(x)}{[n-1]} \sum_{k=0}^{\infty} \frac{1}{[n+k-2]} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} \left(q^2x\right)^k \\ &= \frac{[2]q^{n-1}xP_{n,q}(x)}{[n-1][n-2]} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-3 \\ k \end{bmatrix} \left(q^2x\right)^k \\ &= \frac{[2]q^{n-1}xP_{n,q}(x)}{[n-1][n-2]} \cdot \frac{1}{\prod\limits_{j=0}^{n-3} \left(1 - q^j \cdot q^2x\right)} \\ &= \frac{[2]q^{n-1}}{[n-1][n-2]} x(1-x)(1-qx)(1-q^nx). \end{aligned}$$

The completes the proof of (2.3)–(2.4). □

From (2.4) it follows that $R_{n,q}(x) \to 0$ as $n \to \infty$ for all $x \in [0,1]$. On the order hand, for $0 < q < 1$ we have $[n-1] \to \frac{1}{1-q}$ as $n \to \infty$, so

$$M_{n,q}e_2(x) \to x^2 + x(1-x)(1-q), \qquad \text{as } n \to \infty.$$

Hence $(M_{n,q}e_2)_{n\ge1}$ does not converge to e_2. However, the following estimation turns out to be very useful in what follows:

$$\begin{aligned} \left|M_{n,q}e_2(x) - e_2(x)\right| &\le \frac{x(1-x)(1-q^nx)}{[n-1]} + \frac{[2]q^{n-1}x(1-x)(1-qx)(1-q^nx)}{[n-1][n-2]} \\ &\le \frac{1}{4[n-1]} + \frac{[2]}{4[n-1]} = \frac{2+q}{4[n]} \cdot \frac{[n]}{[n-1]}, \end{aligned}$$

and since

$$\frac{[n]}{[n-1]} = \frac{1-q^n}{1-q^{n-1}} \le 1+q,$$

we finally get

$$\left|M_{n,q}e_2\left(x\right) - e_2\left(x\right)\right| \le \frac{(2+q)(1+q)}{4[n]} \le \frac{3}{2[n]} \tag{2.6}$$

for all integers $n \ge 3$ and all $x \in [0,1]$.

By proceeding like G. M. Phillips [10], in order to obtain a convergent sequence of generalized Meyer-König and Zeller operators, we let $q = q_n$ depend on n. More precisely, we choose a sequence $(q_n)_{n\ge1}$ of real numbers satisfying

$$1 - \frac{1}{n} \le q_n < 1, \qquad \text{for all} \quad n \ge 1. \tag{2.7}$$

Then we have

$$1 - \frac{r}{n} \le q_n^r < 1, \qquad \text{for all} \quad 1 \le r \le n-1,$$

hence

$$[n] = 1 + q_n + q_n^2 + \ldots + q_n^{n-1} \ge n - \frac{n(n-1)}{2n} = \frac{n+1}{2}.$$

Taking into account (2.6) we deduce that

$$\|M_{n,q_n}e_2 - e_2\| \le \frac{3}{n+1}$$

for all $n \ge 3$. Consequently, the sequence $(M_{n,q_n}e_2)_{n\ge1}$ converges uniformly to e_2 on $[0,1]$. By applying the well-known Bohman-Korovkin theorem, we can conclude that the following theorem hods:

Theorem 2.2. *If $(q_n)_{n\ge1}$ is a sequence of real numbers satisfying* (2.7), *then for each $f \in C[0,1]$ the sequence $(M_{n,q_n}f)_{n\ge1}$ converges uniformly to f on* $[0,1]$.

Given a function $f : [0,1] \to \mathbb{R}$ as well as a positive real number δ, let

$$\omega\left(f,\delta\right) = \sup\left\{\, |f(x_1) - f(x_2)| : x_1, x_2 \in [0,1],\ |x_1 - x_2| \le \delta\right\}$$

denote the usual modulus of continuity of f. From (2.6), by means of Theorem 2.2 in [7], we deduce the following Popoviciu-type theorem for the $M_{n,q}$-operators.

Theorem 2.3. *If $f \in C[0,1]$, then for each integer $n \ge 3$ we have*

$$\|M_{n,q}f - f\| \le \frac{5}{2}\omega\Big(f, \frac{1}{\sqrt{[n]}}\Big).$$

Likewise, from (2.6), by means of Theorem 2.4 in [7], we deduce the following Lorentz-type theorem for the $M_{n,q}$-operators.

THEOREM 2.4. *If $f \in C^1[0,1]$, then for each integer $n \geq 3$ we have*

$$\|M_{n,q}f - f\| \leq \tfrac{3+\sqrt{6}}{2\sqrt{[n]}}\omega\Big(f', \tfrac{1}{\sqrt{[n]}}\Big).$$

Since the proofs of Theorem 2.3 and Theorem 2.4 are quite similar with those of Corollary 2.3 and Corollary 2.5, respectively, in [7], we omit them.

3. CONVEXITY AND GENERALIZED MEYER-KÖNIG AND ZELLER OPERATORS

For any real sequence a, finite of infinite, we denote by $v(a)$ the number of strict sign changes in a. Given a function $f:[0,1] \to \mathbb{R}$, let $V(f)$ be the number of sign changes of f in $[0,1]$, i.e.

$$V(f) = \sup v\big(f(x_1), \ldots, f(x_m)\big)$$

where the supremum is taken over all increasing sequences $0 \leq x_1 < \ldots < x_m \leq 1$, for all positive integers m.

An operator L assigning to each function $f:[0,1] \to \mathbb{R}$ the function $Lf : [0,1] \to \mathbb{R}$ is said to be a *variation diminishing operator* (cf. [11]) if

$$V(Lf) \leq V(f), \qquad \text{for all functions} \;\; f:[0,1] \to \mathbb{R}.$$

THEOREM 3.1. *For each positive integer n, the generalized Meyer-König and Zeller operator $M_{n,q}$ is a variation diminishing operator.*

Proof. By means of the well-known Descartes' rule of signs it is easy to prove that if $a = (a_k)_{k\geq 0}$ is a sequence of real numbers such that the power series $\sum_{k\geq 0} a_k x^k$ converges uniformly on $[0,1]$ to a function g, then

$$V(g) \leq v(a).$$

Taking this into account we have

$$\begin{aligned} V(M_{n,q}f) &= V\left(\sum_{k=0}^{\infty} f\left(\tfrac{[k]}{[n+k]}\right) \left[{n+k \atop k}\right] x^k\right) \\ &\leq V\left(\left(f\big(\tfrac{[k]}{[n+k]}\big)\left[{n+k \atop k}\right]\right)_{k\geq 0}\right) \\ &\leq V(f) \end{aligned}$$

for all continuous functions $f:[0,1] \to \mathbb{R}$. This proves the assertion of the theorem. $\square$

Remark 3.1. From (1.5) it follows that $V(M_{n,q}f) \leq V(f)$ for every bounded function $f:[0,1]\to\mathbb{R}$ for which there exists $\lim_{x\nearrow 1} f(x)$. Taking account of (2.1) and (2.2), by the above theorem we deduce that

$$V(M_{n,q}f - p) = V(M_{n,q}(f-p)) \leq V(f-p) \tag{3.1}$$

for every bounded function $f:[0,1]\to\mathbb{R}$ for which there exists $\lim_{x\nearrow 1} f(x)$ and every linear polynomial p. A standard reasoning based on (3.1) (see, for instance, [5], [11], [6]) yields the following theorem. □

Theorem 3.2. *For each positive integer n the following assertions are true:*

1^0 *If $f:[0,1]\to\mathbb{R}$ is an increasing (decreasing) function, then $M_{n,q}f$ is also increasing (decreasing).*

2^0 *If $f:[0,1]\to\mathbb{R}$ is a convex function, then $M_{n,q}f$ is also convex and $M_{n,q}f(x)\geq f(x)$ for all $x\in[0,1]$.*

Like in the case of the classical Meyer-König and Zeller operators, we have the following.

Theorem 3.3. *If $f:[0,1]\to\mathbb{R}$ is a convex function, then for each $x\in[0,1]$ the sequence $(M_{n,q}f(x))_{n\geq 1}$ is non-increasing.*

Proof. The assertion of the theorem being trivially true for $x=1$, we may assume that $0\leq x<1$. Let n be any positive integer. We have

$$\begin{aligned}\frac{M_{n,q}f(x)-M_{n+1,q}f(x)}{P_{n,q}(x)} &= q^{n+1}x\sum_{k=0}^{\infty} f\left(\tfrac{[k]}{[n+k+1]}\right)\begin{bmatrix}n+k+1\\k\end{bmatrix}x^k\\ &-\sum_{k=1}^{\infty} f\left(\tfrac{[k]}{[n+k+1]}\right)\begin{bmatrix}n+k+1\\k\end{bmatrix}x^k\\ &+\sum_{k=1}^{\infty} f\left(\tfrac{[k]}{[n+k]}\right)\begin{bmatrix}n+k\\k\end{bmatrix}x^k.\end{aligned}$$

Using the recurrence formula

$$\begin{bmatrix}n+k+1\\k\end{bmatrix} = q^{n+1}\begin{bmatrix}n+k\\k-1\end{bmatrix}+\begin{bmatrix}n+k\\k\end{bmatrix}$$

in the second sum we get

$$\begin{aligned}&\frac{M_{n,q}f(x)-M_{n+1,q}f(x)}{P_{n,q}(x)} =\\ &= x\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}x^k\Bigg\{q^{n+1}\tfrac{[n+k+1]}{[n+1]}f\left(\tfrac{[k]}{[n+k+1]}\right)-q^{n+1}\tfrac{[n+k+1]}{[n+1]}f\left(\tfrac{[k+1]}{[n+k+2]}\right)\\ &\qquad -\tfrac{[n+k+1]}{[n+1]}f\left(\tfrac{[k+1]}{[n+k+2]}\right)+\tfrac{[n+k+1]}{[k+1]}f\left(\tfrac{[k+1]}{[n+k+1]}\right)\Bigg\}.\end{aligned}$$

As a simple computation shows, the expression between the braces equals to

$$\frac{q^{n+2k+1}}{[n+k+1][n+k+2]}\left[\frac{[k]}{[n+k+1]},\frac{[k+1]}{[n+k+2]},\frac{[k+1]}{[n+k+1]};f\right]\geq 0$$

because f is convex. Consequently, $M_{n,q}f\left(x\right)\geq M_{n+1,q}f\left(x\right).$ □

REFERENCES

[1] Askey, R., *Ramanujan's extensions of the Gamma and Beta functions*, Amer. Math. Monthly, **87**, pp. 346–359, 1980.

[2] Bailey, W. N., Generalized Hypergeometric Series, Hafner, New York, 1972.

[3] Cheney, E. V. and Sharma, A., *Bernstein power series*, Canad. J. Math., **16**, pp. 241–253, 1964.

[4] Cimoca, G. and Lupaş, A., *Two generalizations of the Mayer-König and Zeller operator*, Mathematica (Cluj), **9 (32)**, pp. 233–240, 1967.

[5] Goodman, T. N. T., Oruç, H. and Phillips, G.M., *Convexity and generalized Bernstein polynomials*, Proc. Edinburg Math. Soc., **42**, pp. 179–190, 1999.

[6] Lorentz, G. G., *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.

[7] Lupaş, A. and Müller, M. W., *Approximation properties of the M_n-operators*, Aequationes Math., **5**, pp. 19–37, 1970.

[8] Meyer-König, W. and Zeller, K., *Bernsteinsche Potenzreihen*, Studia Math., **19**, pp. 89–94, 1960.

[9] Oruç, H., Phillips, G.M. and Davis, P.J., *A generalization of the Bernstein polynomials*, Proc. Edinburg Math. Soc., **42,** pp. 403–413, 1999.

[10] Phillips, G. M., *Bernstein polynomials based on the q-integers*, Ann. Numer. Math., **4**, pp. 511–518, 1997.

[11] Schoenberg, I. J., *On variation diminishing approximation methods*, in *On Numerical Approximation*, R. E. Langer (editor), Madison, pp. 249–274, 1959.

Received September 2, 1999.

"Babeş Bolyai" University
Faculty of Mathematics and Computer Science
1, M. Kogălniceanu St.
3400 Cluj-Napoca, Romania
E-mail: ttrif@math.ubbcluj.ro

Added in proof. After the paper had been sent to the typography, the author found out that the sequence of generalized Meyer-König and Zeller operators considered here, had already been introduced and investigated by Luciana Lupaş, *A q-analogue of the Meyer-König and Zeller operator*, Anal. Univ. Oradea, 2, pp. 62-66, 1992. Thus, part of the results in the present paper (Theorem 2.2 and Theorem 3.3) were established for the first time in the previously quoted article by Luciana Lupaş.