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A NOTE ON THE QUADRATIC CONVERGENCE OF THE INEXACT NEWTON METHODS*

EMIL CĂTINAŞ

Abstract. We show that a new sufficient condition for the convergence with q-order two of the inexact Newton iterates may be obtained by considering the normwise backward error of the approximate steps and a result on perturbed Newton methods.

This condition is in fact equivalent to the characterization given by Dembo, Eisenstat and Steihaug.

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1. INTRODUCTION

The inexact Newton (IN) method is given by the algorithm

$$\begin{aligned}
&\text{Choose an initial approximation } y_0 \in D\\
&\text{For } k = 0, 1, \ldots \text{ until "convergence" do}\\
&\qquad \text{Find } s_k \text{ such that } F'(y_k)\, s_k = -F(y_k) + r_k\\
&\qquad \text{Set } y_{k+1} = y_k + s_k,
\end{aligned}$$

and it constitutes a classical model for the practical solving of nonlinear systems $F(y) = 0$ by the Newton method, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. *The error terms (the residuals)* r_k represent the amounts by which the approximate solutions s_k fail to satisfy the exact linear systems $F'(y_k)\, s = -F(y_k)$.

Under certain conditions, the IN iterates converge to a solution y^* of the mentioned nonlinear system, the convergence order being given by the magnitude of the residuals (see [9]).

We are interested in the high convergence orders of the iterates, namely in the convergence with q-order two (for the definitions of the convergence orders see [13, ch.9], and also [16], [15]). The (standard) assumptions on F for this case are the following:

- there exists $y^* \in D$ such that $F(y^*) = 0$;

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- the mapping F is differentiable on a neighborhood of y^*, with the derivative F' Lipschitz continuous at y^*:[1]

$$\|F'(y) - F'(y^*)\| \leq L\|y - y^*\|, \quad \text{for some } L \geq 0, \text{ when } \|y - y^*\| < \varepsilon;$$

- the Jacobian $F'(y^*)$ is nonsingular.

Dembo, Eisenstat and Steihaug proved the following result.

Theorem 1. [9] *Suppose that F obeys the standard assumptions and that for some initial approximation $y_0 \in D$, the sequence $(y_k)_{k\geq 0}$ given by the IN method converges to y^*. Then its q-convergence order is at least two if and only if $\|r_k\| = \mathcal{O}(\|F(y_k)\|^2)$ as $k \to \infty$, or, equivalently,*

$$\frac{\|r_k\|}{\|F(y_k)\|} = \mathcal{O}(\|F(y_k)\|), \quad \text{as } k \to \infty. \tag{1}$$

In our recent paper [1] (see also [3]) we have introduced the inexact perturbed Newton methods, characterizing their convergence orders in terms of perturbations and residuals. In the following analysis we shall consider the perturbed Newton (PN) iterates:

$$\begin{aligned} (F'(y_k) + \Delta_k)\, s_k &= -F(y_k) + \delta_k \\ y_{k+1} &= y_k + s_k, \quad k = 0, 1, \ldots, \quad y_0 \in D. \end{aligned}$$

The matrices Δ_k and the vectors δ_k are some arbitrary perturbations to the linear systems $F'(y_k)\, s = -F(y_k)$, the perturbed linear systems being verified by the exact solutions s_k (it is implicitely assumed that the perturbed Jacobians $F'(y_k) + \Delta_k$ are invertible for $k = 0, 1, \ldots$). We have obtained the following result: [2]

Corollary 2. [1], [8] *Suppose that F obeys the standard assumptions and that for some initial approximation $y_0 \in D$, the sequence $(y_k)_{k\geq 0}$ given by the PN method is well defined and converges to y^*. If*

$$\|\Delta_k\| = \mathcal{O}(\|F(y_k)\|) \text{ and } \|\delta_k\| = \mathcal{O}(\|F(y_k)\|^2), \quad \text{as } k \to \infty,$$

then $(y_k)_{k\geq 0}$ converges with q-order at least two.

In the following section we shall show that the sufficient condition (1) may be expressed in an equivalent form.

[1] We shall use the symbols $\|\cdot\|$ and $\|\cdot\|_2$ for an arbitrary, resp. for the Euclidean norm on $\mathbb{R}^n$, and for their induced operator norms.

[2] The same result can be found in [8]; we have obtained independently this Corollary in the manuscript of [1], before [8] was published.

2. MAIN RESULT

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, $b \in \mathbb{R}^n$ an arbitrary vector, and consider an approximate solution $\tilde{x} \in \mathbb{R}^n$ of the linear system $Ax = b$. The *normwise backward error of* $\tilde{x}$ was introduced by Rigal and Gaches [17] and is defined by:

$$\Pi(\tilde{x}) = \min\left\{\varepsilon : (A + \Delta_A)\,\tilde{x} = b + \Delta_b,\ \left\|\Delta_A\right\|_F \le \varepsilon \left\|E\right\|_F,\ \left\|\Delta_b\right\|_2 \le \varepsilon \left\|f\right\|_2\right\},$$

where the parameters $E \in \mathbb{R}^{n\times n}$ and $f \in \mathbb{R}^n$ are arbitrary, and $\|\cdot\|_F$ denotes the Frobenius norm: $\|Z\|_F = \Big(\sum_{i,j=1,\ldots,n} z_{ij}^2\Big)^{1/2}$. The value of $\Pi(\tilde{x})$ is

$$\Pi(\tilde{x}) = \frac{\|b - A\tilde{x}\|_2}{\|E\|_F \cdot \|\tilde{x}\|_2 + \|f\|_2},$$

and the minimum is attained by the *backward errors*

$$\Delta_A = \frac{\|E\|_F \cdot \|\tilde{x}\|_2}{\|E\|_F \cdot \|\tilde{x}\|_2 + \|f\|_2}\,(b - A\tilde{x})\, z^t, \quad \text{with } z = \tfrac{1}{\|\tilde{x}\|_2^2}\tilde{x},$$

$$\Delta_b = -\frac{\|f\|_2}{\|E\|_F \cdot \|\tilde{x}\|_2 + \|f\|_2}\,(b - A\tilde{x}).$$

With these relations we can prove the following result.

THEOREM 3. *Suppose that F obeys the standard assumptions and that for some initial approximation $y_0 \in D$, the sequence $(y_k)_{k\ge 0}$ given by the IN method converges to y^*. If*

$$\text{(2)} \qquad \frac{\|r_k\|_2}{\|s_k\|_2 + \|F(y_k)\|_2} = \mathcal{O}\left(\|F(y_k)\|_2\right), \quad \textit{as } k \to \infty,$$

then the iterates converge with q-order at least two.

Proof. For each k, consider the normwise backward errors of the approximate steps s_k, choosing the parameters E and f such that $\|E\|_F = \|F(y_k)\|_2$ and $\|f\|_2 = \|F(y_k)\|_2^2$. Corollary 2 and the inequality $\|Z\|_2 \le \|Z\|_F$, true for all $Z \in \mathbb{R}^{n\times n}$ (cf., e.g., [11]), lead to the stated result, provided that the ε's are uniformly bounded. This condition is written as

$$\Pi(s_k) = \frac{\|r_k\|_2}{\|s_k\|_2 \cdot \|F(y_k)\|_2 + \|F(y_k)\|_2^2} \le K, \quad \text{for some fixed } K > 0,$$

which is exactly relation (2). □

REMARKS. 1. Condition (2) is in fact equivalent to condition (1). Indeed, Theorem 3 shows that relation (2) implies the quadratic convergence. For the converse, assuming that the IN iterates converge quadratically, we notice that

$$\frac{\|r_k\|_2}{\|s_k\|_2 + \|F(y_k)\|_2} \le \frac{\|r_k\|_2}{\|F(y_k)\|_2} \le K\,\|F(y_k)\|_2, \quad k = 0, 1, \ldots,$$

which shows that quadratic convergence implies (2) (the right inequality above is ensured by Theorem 1).

2. It is easy to prove that the Euclidean norm from (2) may be replaced by an arbitrary norm on $\mathbb{R}^n$, since all norms are equivalent on a finite dimensional normed space.

3. Condition (2) has been obtained in a natural fashion by considering the normwise backward errors of the approximate steps. One can also obtain it taking into account that under the standard assumptions, the sequences $(y_k - y^*)_{k\geq 0}$ and $(s_k)_{k\geq 0}$ converge quadratically to zero only at the same time, with $\lim_{k\to\infty} \|y_k - y^*\| / \|y_{k+1} - y_k\| = 1$ (see [18]). The same situation appears concerning $(y_k - y^*)_{k\geq 0}$ and $(F(y_k))_{k\geq 0}$, since the standard assumptions ensure that there exists $\alpha > 0$ such that $\|y - y^*\| / \alpha \leq \|F(y)\| \leq \alpha \|y - y^*\|$ when y is sufficiently close to y^* (see [9]). □

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"T. Popoviciu" Institute of Numerical Analysis
Romanian Academy
P.O. Box 68, 3400 Cluj-Napoca 1
Romania