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A UNIFIED TREATMENT OF BOUNDARY LAYER AND LUBRICATION APPROXIMATIONS IN VISCOUS FLUID MECHANICS

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Abstract. It is a matter of every day experience to find the boundary layer and lubrication approximations exposed as if they had nothing in common. It is the aim of this note to show that, in fact, they come from the Navier-Stokes system and that they correspond to some distinguished limits of the Reynolds number.

Key words: boundary layer, lubrication, Reynolds number, distinguished limits.

1. INTRODUCTION

lot of literature does exist concerning the concepts of viscous flows in boundary layers and in thin layers (films). Sometimes, a rather confusing situation may occur when people try to consider flows such that viscous effects are important in thin layers. One could ask: what is the ground on which it is possible to choose between these two approximatons? Only the conventional wisdom or empirical data?

It is the goal of this approach to underline the idea that boundary layer approximation and lubrication approximation, called also thin layer (film) theory, correspond to two distinguished limits of Reynolds number in Navier-Stokes system, provided that, the thickness δ of boundary layer and thin layer are comparable (have the same order of magnitude).

We will use the method of dominant balance in order to reach these distinguished limits. The method is available in the important work of Bender & Orszag [2]. A lot of results obtained using this method are scattered in a huge number of articles and in some books devoted to the so-called applied mathematics. We refer here to Lin and Segel [9], Fowkes and Mahony [3] and Fowler [4], to quote but a few.

There are also many books on fluid mechanics which pay attention to these concepts.

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The book of Schlichting [12] is a classic and voluminous engineering approach to boundary layer theory. In the Romanian literature the same is Oroveanu [11].

For the lubrication theory we should mention the book of Acheson [1], which is a nice introduction not only in this topic, and the book of Ockendon and Ockendon [10] which is a short gloss with some novel applications and insights.

2. BOUNDARY LAYER APPROXIMATION VS. LUBRICATION APPROXIMATION

If a typical flow velocity is $\overline{U}$, and a typical flow geometry is of dimension l, then if we nondimensionalize the variables as follows:

$$\mathbf{u} \sim \overline{U},\ \mathbf{x} \sim l,\ p - p_\infty \sim p\overline{U}^2, t \sim l\,/U$$

i.e. write $u = \overline{U}u*$, etc., substitute in Navier-Stokes system and drop asterisks, we get

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$u_t + (\mathbf{u} \cdot \nabla)\, u = -\nabla p + \frac{1}{\mathrm{Re}} \cdot \nabla^2 \mathbf{u}.$$

The dimensionless parameter Re is $\mathrm{Re} = \overline{U}l/\nu$, and the sign $\sim$ stands for asymptotic equivalent.

Here p_∞ is an ambient pressure, for example, at infinity.

Let us consider now a more specific situation, namely a two dimensional configuration, where the transverse y length scale, denoted δ, relative to the longitudinal x one, denoted l, is small. The fluid to be taken into account is viscous and incompressible. Let us denote by ε the aspect ratio, i.e. $\varepsilon := \delta/l$, and consider $0 < \varepsilon \ll 1$.

We have to remark that from the mathematical point view the parameter ε has an asymptotic meaning.

With respect to the boundary layer concept, a lot of empirical (experimental) as well as numerical estimations for δ are available.

Lubrication approximation, on the other hand, is frequently used in physico-chemical hydrodynamics. It is typical there, that the thickness δ of layers where viscous effects are important is small compared to the dimension of the bulk flow. It is well suited to quote here, beyond the classics Levich [8], or Landau and Lifshitz [7] (Ch.VII, Surface phenomena), our contributions [5], [6] where flows driven by surface tension gradients are considered and the recent study of Wilson, Davis and Bankoff [13].

Thus, if we rescale the Navier-Stokes system (1) by writing

$$y \sim \varepsilon,\ v \sim \varepsilon,\ p \sim 1/\varepsilon^2\mathrm{Re} \tag{2}$$

with $\mathbf{u}= (u, v)$, these equations become:

$$\begin{aligned} &u_x + u_y = 0,\\ &\varepsilon^2 \mathrm{Re}\,(du/dt) = -p_x + u_{yy} + \varepsilon^2 u_{xx},\\ &\varepsilon^4 \mathrm{Re}\,(du/dt) = -p_y + \varepsilon^2\left(v_{yy} + \varepsilon^2 v_{xx}\right). \end{aligned} \tag{3}$$

Consequently, if Reynolds number does not exceed a quantity of order $0\left(\frac{1}{\varepsilon}\right)$, the system (3) leads to Reynolds' equations of lubrication theory, which determines the flow solution given δ and a flow rate (flux) (see for example Fowler [4], Ch. 6, Viscous flow).

If Reynolds number becomes larger, more exactly, $\mathrm{Re} = 0\left(\frac{1}{\varepsilon^2}\right)$, the same system (3) produces the boundary layer equations, i.e.

$$\begin{aligned} &u_x + v_y = 0,\\ &uu_x + vu_x = u_{yy} + U \cdot U_x, \end{aligned} \tag{4}$$

where $U(x) := u(x, \infty)$.

It is now clear that the two distinguished limits of Reynolds number corresponding to lubrication approximation and boundary layer approximation are respectively $1/\varepsilon$ and $1/\varepsilon^2$.

It is also worth noting that, the only way the second equation in (3) will not reduce to a triviality as $\varepsilon \to 0$ is the case when the pressure is rescaled with $1/\varepsilon^2$. This is why in (3) we choose the scale $1/\varepsilon^2\mathrm{Re}$ for pressure. With this rescaling the pressure field becomes $p = 0(1)$ in boundary layer approximation and $p = 0\,(1/\varepsilon)$, $0 < \varepsilon \ll 1$ in lubrication approximation.

3. CONCLUDING REMARKS

The above discussion is independent with respect to various boundary conditions that would be specified for boundary layer equations (4) or lubrication equations. A no-slip boundary condition on a solid boundary is common for both approximations. In the first one, conditions for velocity and pressure away from the solid boundary and in the later, shear stress conditions, kinematic boundary conditions or even Navier slip conditions on interfaces, are to be imposed to close the differential system.

Unfortunately, our analysis restricts itself to two dimensional flows.

Last but not least, we have to observe that lubrication theory is a linear one, or in other words, even at high Reynolds numbers, when the flow is shallow, the inertia (convective) terms are negligible. For higher Reynolds numbers, the boundary layer system remains essentially nonlinear.

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