REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 29 (2000) no. 1–2, pp. 139–149
ictp.acad.ro/jnaat

ON THE NUMERICAL EVALUATION OF CERTAIN 2-D SINGULAR INTEGRALS

LAURA GORI, LAURA LO CASCIO and ELISABETTA SANTI

Abstract. The problem of approximating certain two-dimensional Cauchy principal value integrals is here considered, product integration formulas with multiple nodes are presented and the behaviour of the remainder is analyzed. Next a particular class of cubature rules is generated, having the peculiar property that the set of the nodes holds fixed while their multiplicities vary. Some numerical examples of application of the latter rules are also provided.

MSC 2000. 65D30, 65D32.

Keywords. product formulas, singular integrals, s-orthogonal polynomials.

1. INTRODUCTION

It is well known that the numerical evaluation of the following two-dimensional Cauchy principal value (CPV) integrals

$$(1)\qquad I(Wf;\xi,\eta)=\fint_{-1}^{1}\fint_{-1}^{1}W(x,y)\,\tfrac{f(x,y)}{(x-\xi)\,(y-\eta)}\,dx\,dy,\qquad |\xi|<1,\ |\eta|<1$$

is of interest in many applications; the approaches for approximating (1) are of two types: local and global. Local methods based on the use of splines have been considered, for instance in [2]; they are suitable, in particular, when f is not smooth, while, when f is differentiable, global methods are probably to be preferred. In this concern, product formulas based on the use of orthogonal polynomials have been proposed (see, for instance, [8], [10]); it is the case of formulas having the form

$$(2)\qquad \fint_{-1}^{1}\fint_{-1}^{1}p_1(x)\,p_2(y)\,\tfrac{f(x,y)}{(x-\xi)\,(y-\eta)}\,dx\,dy\cong\sum_{i=1}^{m}\sum_{j=1}^{n}C_{ij}(\xi,\eta)\,f(x_i,y_j)$$
$$:=Q_{m,n}(f;\xi,\eta)\,.$$

This work was supported by MURST of Italy.

Here p_1 and p_2 are Jacobi weight functions, the rules are of interpolatory type, $\{x_i\}_{i=1}^m$, $\{y_j\}_{j=1}^n$ are the zeros of orthogonal polynomials relative to p_1 and p_2 respectively, or of Chebyshev polynomials of the first or second kind.

The question of the convergence of $\{Q_{m,n}(f;\xi,\eta)\}$ for $m,\ n$ diverging, was also dealt with in the quoted papers. Yet, very few numerical examples for the evaluation of (2) are provided in the literature, while one of the goals of this paper is to provide several numerical tests showing the behaviour of the cubature rules here proposed.

One of the difficulties which occur in handling (2), is due to the fact that an augment of the precision degree in (1) requires to increase the number of the nodes: this implies a twofold disadvantage, because not only all the nodes must be evaluated again, but also numerical cancellation may occur due to the decreasing of $\min_i |x_i - \xi|$ and/or $\min_j |y_j - \eta|$, although the case $x_i \neq \xi,\ y_j \neq \eta,\ i = 1, ..., m,\ j = 1, ..., n$ is always assumed to hold.

Another general approach, which enables one to increase the precision degree of an integration rule, amounts to construct formulas of Turán type, that is having multiple nodes with odd multiplicities, say $2s + 1$; in which case a higher precision is attainable by a higher multiciplity of the nodes. The use of Turán quadrature rules for approximating one-dimensional CPV integrals has been developed in [7]. However, also in the case of rules based on multiple nodes, a change in the multiplicity generally implies a change of the zeros of the s-orthogonal polynomials involved in the construction; thus, the just mentioned computational problems may occur, as well.

For this reason the introduction of formulas with multiple nodes assumes a particular value in those cases in which the nodes are independent of the multiplicity, so that the computational problems quoted before can be avoided and the corresponding formulas are really effective.

In this context, are crucial some results concerning an invariance property of some classes of s-orthogonal polynomials, which are briefly summarized in Section 2. In Section 3, a general cubature rule for approximating the two-dimensional CPV integrals (1) is given and an expression of the remainder is provided. In Section 4, the results of [6] are exploited in order to construct certain formulas such that the cubature sum and the remainder have particularly interesting features, which, among other things, allow one to deduce the asymptotic behaviour of the remainder. Finally, Section 5 is devoted to the development of some examples, which put in evidence the good numerical performances of the integration formulas based on the zeros of s-orthogonal polynomials enjoying a "s-invariance" property.

2. PRELIMINARIES

We recall that, given on the real line, an interval A, finite or infinite, a weight function w, satisfying the condition $w(x) \geq 0, \forall x \in A$ and such that all the moments

$$m_i = \int_A w(x)\ x^i\, dx, \qquad i = 0, 1, 2 \ldots$$

exist finite (in particular $m_0 > 0$), and given a nonnegative integer s, the monic polynomials of the sequence $\{P_{ms}(w;\bullet)\}_{m\in\mathbb{N}}$, $P_{ms} \in \mathbb{P}_m$ ($\mathbb{N}$ is the set of natural numbers and $\mathbb{P}_m$ denotes the set of algebraic polynomials of degree m) are said to be s-orthogonal in A with respect to w, if they are the polynomials minimizing the w-weighted L_{2s+2} norm [4, pg. 75], i.e.

$$\int_A w(x)\left[P_{ms}(w;x)\right]^{2s+2} dx = \text{minimum}.$$

This minimization leads to the conditions

$$\int_A w(x)x^k \left[P_{ms}(w;x)\right]^{2s+2} dx = 0, \qquad k = 0, 1, \ldots, m-1;$$

every P_{ms} has m zeros $\{x_i^s\}_{i=1}^m$ real and simple in the interior of A.

A well known result of Bernstein [3] shows that, when w is the Chebyshev weight function of the first kind, the polynomials of minimal weighted L_p norm, for any $p \in [1, +\infty)$, are the corresponding Chebyshev polynomials

$$\{T_m(x) = \cos m\,(\arccos\, x)\}_{m\in\mathbb{N}}\,;$$

this means that the sequence of s-orthogonal polynomials is independent of s.

More recently, some other cases of invariance with respect to s have been considered in [5] and [6]. In [5] it was introduced a particular class of weight functions w_μ, depending on a real parameter $\mu > -1$ such that the polynomials of second degree, s-orthogonal in $[-1, 1]$ with respect to w_μ, are invariant for any s and any μ. In [6], this result was extended to polynomials of any degree n, identifying a wide class W_n of weight functions, enjoying an analogous invariance property.

This class, containing in particular the weight functions w_μ, is characterized as follows: let w_∞ denote the Chebyshev weight function of the first kind; a weight function w_n is said to belong to W_n if it fulfills the relation

$$\frac{w_n(x)}{w_\infty(x)} = \sum_{l=0}^{\infty} \rho_l T_{2nl}(x)\,.$$

In fact, in [6] it was proved that, for any given n and any $w_n \in W_n$, the polynomial T_n satisfies the following condition

$$\min_{p_{n-1}\in\mathbb{P}_{n-1}} \left\{ \int_{-1}^{1} w_n(x) \left|T_n(x) - p_{n-1}(x)\right|^{\gamma} dx, \quad p_{n-1} \in \mathbb{P}_{n-1} \right\} = \\ = \int_{-1}^{1} w_n(x) \left|T_n(x)\right|^{\gamma} dx$$

where $\gamma \in \Re, \gamma \geq 1$. Thus, assuming $\gamma = 2s + 2$, it turns out that T_n is s-orthogonal in $[-1, 1]$ with respect to w_n, independent of s.

A subset $W_{n,\mu}$ of W_n is provided by the functions $w_{n,\mu}$ defined by:

$$w_{n,\mu}(x) = \left|\tfrac{U_{n-1}(x)}{n}\right|^{2\mu+1} \left(1 - x^2\right)^{\mu}, \qquad x \in [-1, 1], \mu > -1, \tag{3}$$

where $U_k(x) = \frac{\sin[(k+1)\arccos x]}{\sqrt{1-x^2}}$ is the Chebyshev polynomial of the second kind.

We observe that all the weight functions in $W_{n,\mu}$ are generalized smooth Jacobi weights [9], and assuming $\mu = -\frac{1}{2}$ one finds out the mentioned result of Bernstein.

3. THE TURÁN TYPE INTEGRATION RULES FOR 2-D CPV INTEGRALS

Let us recall that the general quadrature rule of Turán type is given by

$$\int_{-1}^{1} p(x) F(x) dx = \sum_{i=1}^{m} \sum_{h=0}^{2s} A_{hi} F^{(h)}\left(x_i^{(s)}\right) + r_s(F) \\ := Q_{ms}(F) + r_s(F) \tag{4}$$

where $\left\{x_i^{(s)}\right\}_{i=1}^{m}$ are the zeros of the m-th degree polynomials $P_{ms}(p; x)$, monic, s-orthogonal with respect to p.

Moreover, we use the notation below for the one-dimensional CPV integral:

$$I(pf; \zeta) = \fint_{-1}^{1} p(x) \tfrac{f(x)}{x-\zeta} dx; \tag{5}$$

subtracting out the singularity one has:

$$I(pf; \zeta) = \int_{-1}^{1} p(x) \tfrac{f(x)-f(\zeta)}{x-\zeta} dx + f(\zeta) \fint_{-1}^{1} \tfrac{p(x)}{x-\zeta} dx$$

and applying (4) to the first integral in the right hand side, yields [7]

(6) $$⨍_{-1}^{1} p(x)\,\frac{f(x)}{x-\zeta}dx = f(\zeta)C_s(\zeta) + \sum_{i=1}^{m}\sum_{h=0}^{2s} B_{hi} f^{(h)}\left(x_i^{(s)}\right) + e_s(f)$$

where

$$C_s(\zeta) = ⨍_{-1}^{1} \frac{p(x)}{x-\zeta}dx - Q_{ms}\left(\frac{1}{x-\zeta}\right)$$

$$B_{hi} = \sum_{k=h}^{2s}(-1)^{k-h}\binom{k}{h}\frac{(k-h)!}{\left(x_i^{(s)}-\zeta\right)^{k-h+1}}A_{ki}$$

$$e_s(f) = r_s\left(\frac{f(x)-f(\zeta)}{x-\zeta}\right).$$

In particular, it has been proved in [7], that if $f \in C^{M+1}[-1,1], M = m(2s+2)$, then

(7) $$e_s(f) = \frac{f^{M+1}(\widetilde{x})}{(M+1)!}\int_{-1}^{1} p(x)\left[P_{ms}(p;x)\right]^{2s+2}dx, \qquad \widetilde{x} \in (-1,1).$$

Now, considering the two-dimensional CPV integral (1) where $W(x,y) = p_1(x)p_2(y)$, and applying a product of quadrature rules (6), we have the general integration rule for approximating (1):

(8)

$$\begin{aligned} I(Wf;\xi,\eta) = & \sum_{h=0}^{2s}\sum_{i=1}^{m}\sum_{k=0}^{2s}\sum_{j=1}^{n} B_{hi}^{(1)}B_{kj}^{(2)} f_h^k\left(x_i^{(s)}, y_j^{(s)}\right) \\ & + C_1(\xi)\sum_{k=0}^{2s}\sum_{j=1}^{n} B_{kj}^{(2)} f_k^0\left(\xi, y_j^{(s)}\right) + C_2(\eta)\sum_{h=0}^{2s}\sum_{i=1}^{m} B_{hi}^{(1)} f_0^h\left(x_i^{(s)}, \eta\right) \\ & + C_1(\xi)C_2(\eta)f(\xi,\eta) + E_s(f) \end{aligned}$$

where $f_h^k = \frac{\partial^{h+k}f}{\partial x^h \partial y^k}$ and

$$B_{hi}^{(1)} = \sum_{p=h}^{2s}\binom{p}{h}\left(D^{p-h}\frac{1}{x-\xi}\right)_{x=x_i} A_{pi}^{(1)},$$

$$B_{kj}^{(2)} = \sum_{q=k}^{2s}\binom{q}{k}\left(D^{q-k}\frac{1}{y-\eta}\right)_{y=y_j} A_{qj}^{(2)},$$

$$C_1(\xi) = ⨍_{-1}^{1}\frac{p_1(x)dx}{x-\xi} - \sum_{h=0}^{2s}\sum_{i=1}^{m} A_{hi}^{(1)}\left(D^h\frac{1}{x-\xi}\right)_{x=x_i},$$

$$C_2(\eta) = ⨍_{-1}^{1}\frac{p_2(y)dy}{y-\eta} - \sum_{k=0}^{2s}\sum_{j=1}^{n} A_{kj}^{(2)}\left(D^k\frac{1}{y-\eta}\right)_{y=y_j}.$$

Here $\left\{x_i^{(s)}\right\}_{i=1}^{m}, \left\{y_j^{(s)}\right\}_{j=1}^{n}$ are the zeros respectively of the s-orthogonal polynomials $P_{ms}(p_1;x)$ and $P_{ns}(p_2;y)$; $A_{\bullet i}^{(1)}$ and $A_{\bullet j}^{(2)}$ are the coefficients of the Turán type quadrature formulas related to the weights p_1 and p_2 respectively.

The error term $E_s(f)$ in (8), vanishes if f is a polynomial of degree M in x and N in y, where

$$M = m(2s+2), \qquad N = n(2s+2).$$

In the previous formulas, the existence of the derivatives of f up to the $2s$-th order is required at least at the nodes. However, the presence of these derivatives in many practical situations does not constitute a computational problem, since in several cases recurrence relations can be established between any two successive derivatives of the given function [1].

In order to give an explicit expression for the error term in (8), we introduce the notation below:

$$D = [-1,1] \times [-1,1],$$
$$m_{0,j} = \int_{-1}^{1} p_j(t)\,dt, \qquad m_{-1,j} = \fint_{-1}^{1} p_j(t)(t-\varsigma)^{-1}\,dt, \qquad j = 1,2,$$

and

(9)
$$H_l(p_j) = \tfrac{1}{(L+1)!}\int_{-1}^{1} p_j(x)\left[P_{ls}(p_j;x)\right]^{2s+2} dx, \quad L = l(2s+2), \qquad j = 1,2.$$

We assume that the quantities $m_{-1,j}(\varsigma)$ exist finite for $j = 1,2$.

THEOREM 1. *If $f \in C^{M+N+2}(D)$, the remainder term in (8) can be expressed in the form:*

$$\begin{aligned} E_s(f) = & H_m(p_1)\left[m_{0,2} f_1^{M+1}(x',y') + m_{-1,2}(\eta)\, f_0^{M+1}(x',\eta)\right] \\ & + H_n(p_2)\left[m_{0,1} f_{N+1}^{1}(x^*,y^*) + m_{-1,1}(\xi)\, f_{N+1}^{0}(\xi,y^*)\right] \\ & - H_m(p_1)\, H_n(p_2)\, f_{N+1}^{M+1}(\sigma,\tau), \end{aligned} \tag{10}$$

where $x', x^*, y', y^*, \sigma, \tau$ belong to $(-1,1)$.

Proof. We recall that an interesting result in [12] allows for giving evaluations of the remainder of some approximation formulas in two variables; to be more precise, let T_1, T_2 denote two linear approximation operators and set $T = T_2(T_1(\bullet))$; if R_1, R_2 and R denote the remainders in the approximation of T_1, T_2 and T respectively, then the following relation holds:

$$R(T(f)) = R_1(T_2(f)) + R_2(T_1(f)) - R_2(R_1(f)).$$

Taking into account that in the case being examinated the linear operators T_1, T_2 reduce to the integral operators $I(p_1 f;\xi)$, $I(p_2 f;\eta)$ defined in (5) and the corresponding error terms have the form (7), we can write

$$\begin{aligned} E_s(f) =& H_m(p_1) \,⨍_{-1}^{1} p_2(y) \tfrac{f_0^{M+1}(x',y)}{y-\eta} dy \\ &+ H_n(p_2) \,⨍_{-1}^{1} p_1(x) \tfrac{f_{N+1}^{0}(x,y^{+})}{x-\xi} dx \\ &- H_m(p_1) H_n(p_2) f_{N+1}^{M+1}(\sigma,\tau). \end{aligned}$$

Subtracting out the singularities in the above CPV integrals and by the hypothesis on f, the relation (10) follows. □

4. ON THE EFFICIENCY OF A CLASS OF INTEGRATION RULES

The evaluation of both the cubature sum and the remainder term in (8) becomes particularly simple when p_1 and p_2 belong to the class of the weight functions recalled in Section 2; indeed, assuming for instance $p_1 \in W_m$ and $p_2 \in W_n$ not only one has

$$x_i^{(s)} = \cos \tfrac{2i-1}{2m}\pi, \quad i = 1, \ldots, m, \qquad y_j^{(s)} = \cos \tfrac{2j-1}{2n}\pi, \quad j = 1, \ldots, n,$$

for any s, but also the coefficients A_{hi} in (4) can be given explicitly, as shown in [6].

Furthermore, in this case, it is possible to state a convergence result of the integration rules (8), for s diverging.

Let $DT(J)$ be the set of Dini type functions, defined on any interval J of length $l(J)$, by

$$DT(J) := \left\{ g \in C(J) : \int_0^{l(J)} \omega(g;t)\, t^{-1} dt < \infty \right\},$$

where $\omega(g;\bullet)$ denotes the modulus of continuity of the function g; and, for some $\delta > 0$, let us consider the subinterval $N_\delta(\varsigma) = [\varsigma - \delta, \varsigma + \delta] \subset [-1, 1]$.

We denote by $t_{i,j}$ the singularities of $p_j,\ i = 1, 2, ..., q_j,\ j = 1, 2$ belonging to $(-1, 1)$ and with U_j the set

$$U_j = \bigcup_{i=0}^{q_j} [a_i, b_i], \quad [a_i, b_i] \subset [t_{i,j}, t_{i+1,j}] \quad \forall i = 0, 1, ..., q_j$$

where $t_{0,j} = -1, t_{q_j+1,j} = 1$.

If the singularities ξ, η are such that $\xi \in U_1$ and $\eta \in U_2$ then,

$$p_j \in L_1[-1,1] \cap DT(N_\delta(\varsigma)), \quad \varsigma = \begin{cases} \xi & \text{if } j=1 \\ \eta & \text{if } j=2 \end{cases}$$

and the associated values $m_{-1,j}(\varsigma)$ are finite (see, for instance, [11]).

Theorem 2. *Let $f \in C^\infty(D)$, $\left|f_h^k(X)\right| \le V_h^k$ for $X \in D$ and assume*

$$\lim_{s\to\infty} \frac{V_j^{M+1}}{(M+1)!2^{(m-1)(2s+2)}} = 0,$$
$$\lim_{s\to\infty} \frac{V_{N+1}^{j}}{(N+1)!2^{(n-1)(2s+2)}} = 0, \quad j = 0,1;$$
$$\lim_{s\to\infty} \frac{V_{N+1}^{M+1}}{(M+1)!(N+1)!2^{(m+n-2)(2s+2)}} = 0,$$

then

$$\lim_{s\to\infty} E_s(f) = 0.$$

Proof. By (9) and for the assumption on p_1 and p_2, there results

$$|H_l(p_j)| = \frac{1}{(L+1)!}\left|\int_{-1}^{1} p_j(x)\frac{[T_l(x)]^{2s+2}}{2^{(l-1)(2s+2)}}dx\right| \le \frac{m_{0,j}}{(L+1)!2^{(l-1)(2s+2)}}, \quad j=1,2.$$

Next the following estimate yields

$$|E_s(f)| \le \frac{K_{0,2}V_1^{M+1}+K_{-1,2}V_0^{M+1}}{(M+1)!2^{(m-1)(2s+2)}} + \frac{K_{0,1}V_{N+1}^1+K_{-1,1}V_{N+1}^0}{(N+1)!2^{(n-1)(2s+2)}} + \frac{KV_{N+1}^{M+1}}{(M+1)!(N+1)!2^{(m+n-2)(2s+2)}}$$

where $K_{h,l} = m_{0,l}m_{h,j}$, $h = 0,-1$, $l = 1,2$, $j = 1,2$ and $l \neq j$, $K = m_{0,1}m_{0,2}$. Therefore, by the above hypotheses, the claim follows. □

Moreover, it is also interesting to point out that if p_1 and p_2 belong to the subset $W_{n,\mu}$ of W_n, then the terms (9) involved in the expression (10) of the remainder, can be often evaluated in closed form. For instance, when $m = 2$, there results

$$H_2(w_{2,\mu}) = \frac{1}{(4s+5)!2^{2s+2}}\int_{-1}^{1}|x|^{2\mu+1}\left(1-x^2\right)^\mu [T_2(x)]^{2s+2}dx = \frac{1}{(4s+5)!2^{2s+2\mu+3}}B\left(s+\frac{3}{2},\mu+1\right).$$

When $m = 3$ and 2μ is a nonnegative integer, then

$$H_3(w_{3,\mu}) = \frac{1}{(6s+7)!3^{2\mu+1}4^{2s+2}}\int_{-1}^{1}|U_2(x)|^{2\mu+1}\left(1-x^2\right)^\mu [T_3(x)]^{2s+2}dx = \frac{2}{(6s+7)!3^{2\mu+1}4^{2s+2}}\frac{(2\mu)!!(2s+1)!!}{(2s+2\mu+3)!!}.$$

In order to give an idea of the magnitude of the terms H_m in (9), we report in the Tables 4.1 - 4.3 some values of $H_m\left(w_{m,\mu}\right)$ corresponding to some weights of type (3), for given values of μ and m.

$m=2$	$\mu=-1/2$	$\mu=0$	$\mu=1/2$	$\mu=1$
$s=0$	$0.163\cdot10^{-01}$	$0.347\cdot10^{-02}$	$0.102\cdot10^{-02}$	$0.347\cdot10^{-03}$
$s=1$	$0.183\cdot10^{-05}$	$0.310\cdot10^{-06}$	$0.761\cdot10^{-07}$	$0.221\cdot10^{-07}$
$s=2$	$0.320\cdot10^{-10}$	$0.466\cdot10^{-11}$	$0.100\cdot10^{-11}$	$0.259\cdot10^{-12}$
$s=3$	$0.161\cdot10^{-15}$	$0.208\cdot10^{-16}$	$0.402\cdot10^{-17}$	$0.944\cdot10^{-18}$

Table 1.

$m=3$	$\mu=-1/2$	$\mu=0$	$\mu=1/2$	$\mu=1$
$s=0$	$0.136\cdot10^{-03}$	$0.193\cdot10^{-04}$	$0.379\cdot10^{-05}$	$0.857\cdot10^{-06}$
$s=1$	$0.960\cdot10^{-11}$	$0.109\cdot10^{-11}$	$0.178\cdot10^{-12}$	$0.345\cdot10^{-14}$
$s=2$	$0.374\cdot10^{-19}$	$0.363\cdot10^{-20}$	$0.520\cdot10^{-21}$	$0.896\cdot10^{-22}$
$s=3$	$0.211\cdot10^{-28}$	$0.182\cdot10^{-29}$	$0.231\cdot10^{-30}$	$0.368\cdot10^{-31}$

Table 2.

$m=4$	$\mu=-1/2$	$\mu=0$	$\mu=1/2$	$\mu=1$
$s=0$	$0.609\cdot10^{-06}$	$0.647\cdot10^{-07}$	$0.952\cdot10^{-08}$	$0.162\cdot10^{-08}$
$s=1$	$0.138\cdot10^{-16}$	$0.117\cdot10^{-17}$	$0.144\cdot10^{-18}$	$0.209\cdot10^{-19}$
$s=2$	$0.604\cdot10^{-29}$	$0.439\cdot10^{-30}$	$0.472\cdot10^{-31}$	$0.610\cdot10^{-32}$
$s=3$	$0.195\cdot10^{-42}$	$0.126\cdot10^{-43}$	$0.122\cdot10^{-44}$	$0.143\cdot10^{-45}$

Table 3.

5. NUMERICAL RESULTS

The integration rules presented in Section 3 have been tested for several functions and for different choices of the weight functions of type (3). We emphasize the very good performances of such cubatures, when the weight functions are $p_1 \in W_{m,\mu_1},\ p_2 \in W_{n,\mu_2}$ as the following Tables illustrate.

- Table 1 refers to the function $f\left(x,y\right)=e^{3x-y}$, assuming $p_1=p_2=w_{m,-1/2}=w_\infty,\ \xi=-0.2,\ \eta=0.5$; the results obtained assuming $m=n=2$ and $m=n=3$ are compared.
- Table 2 shows the results when $f\left(x,y\right)=\frac{e^x}{y^2+25}$, assuming $p_1=w_{m,0},\ p_2=w_{n,-1/2}$ and $m=2,\ n=3$ for two different positions of the singular point $\left(\xi,\eta\right)$.

Finally the case of oscillating functions is considered:

- Table 3 refers to the function $f\left(x,y\right)=\sin\left(x+y\right)$, the weight functions are $p_1=w_{m,-1/2},\ p_2=w_{m,0}$, the singularity is $\left(\xi,\eta\right)=\left(-0.1,0.25\right)$; we consider the values $m=n=2$ and $m=2,\ n=3$.

- Table 4 shows the results for the function $f(x,y) = \sin(2x+3y)$, when p_1 and p_2 are as in the previous example, $m = n = 2$ and $(\xi,\eta) = (-0.95, 0.90)$.

s	$m=n=2$	$m=n=3$
0	-40.1796987881	-42.7919904372
1	-42.9178803324	-42.9304171601
2	-42.9304088530	-42.9304207223
3	-42.9304207187	
4	-42.9304207223	

Table 4. $f(x,y) = e^{3x-y},\ \xi = -0.2,\ \eta = 0.5$

s	$\xi = 0,\ \eta = 0.25$	$\xi = 0,\ \eta = 0.99$
0	$-1.334\,185\,174\,340 \cdot 10^{-03}$	$-5.096\,768\,029\,771 \cdot 10^{-03}$
1	$-1.335\,067\,155\,027 \cdot 10^{-03}$	$-5.100\,137\,315\,429 \cdot 10^{-03}$
2	$-1.335\,067\,198\,346 \cdot 10^{-03}$	$-5.100\,137\,480\,914 \cdot 10^{-03}$
3	$-1.335\,067\,198\,347 \cdot 10^{-03}$	$-5.100\,137\,480\,915 \cdot 10^{-03}$

Table 5. $f(x,y) = \dfrac{e^x}{y^2+25},\ m = 2, n = 3$

s	$m=n=2$	$m=2,\ n=3$
0	$-2.067\,887\,525\,787 \cdot 10^{-01}$	$-6.812\,202\,145\,283 \cdot 10^{-01}$
1	$-2.070\,973\,210\,785 \cdot 10^{-01}$	$-6.815\,485\,070\,316 \cdot 10^{-01}$
2	$-2.070\,973\,402\,908 \cdot 10^{-01}$	$-6.815\,485\,220\,557 \cdot 10^{-01}$
3	$-2.070\,973\,402\,910 \cdot 10^{-01}$	$-6.815\,485\,220\,558 \cdot 10^{-01}$

Table 6. $f(x,y) = \sin(x+y),\ \xi = -0.1,\ \eta = 0.25$

s	$m=2,\ n=2$
0	-4.639773465749
1	-4.232307580999
2	-4.230525126503
3	-4.230525126044

Table 7. $f(x,y) = \sin(2x+3y),\ \xi = -0.95,\ \eta = 0.90$

We point out that also in the case when the singularity is close to the boundary of the integration interval, the rules here proposed have a good performance.

REFERENCES

[1] R.A. Cicenia, *Numerical integration formulas involving derivatives*, J. Inst. Math. Appl. **18** (1976), 79-85.

[2] C. Dagnino, S. Perotto and E. Santi, *Product formulas based on spline approximation for the numerical evaluation of certain 2-D CPV integrals,* Approx. And Optim. (Eds. D.D. Stancu, G. Coman, W.W. Breckener, P. Blaga), Transilvania Press, Cluj-Napoca (1997), 241-250.

[3] W. Gautschi, *A survey of Gauss-Christoffel quadrature formulae*, G.B. Christoffel (P.L. Butzer and F. Fehèr eds.) Birkhäuser, Basel (1981), 72-147.

[4] A. Ghizzetti, A. Ossicini, *Quadrature formulae*, Birkhäuser, Basel 1970.

[5] L. Gori, M.L. Lo Cascio, *On an invariance property of the zeros of some s-orthogonal polynomials* , Orthogonal polynomials and their applications (Eds. C. Brezinski, L. Gori, A. Ronveaux) J.C. Baltzer AG Scient. Publ. IMACS (1991), 277-280.

[6] L. Gori, C.A. Micchelli, *On weight functions which admit explicit Gauss-Turán quadrature formulas*, Math Comp. **65**(1996), 1567-1581.

[7] L. Gori, E. Santi, *On the evaluation of Hilbert transform by means of a particular class of Turán type quadrature rules*, Numer. Algorithms **10** (1995), 27-39.

[8] G. Monegato, *Convergence of product formulas for the numerical evaluation of certain two-dimensional Cauchy principal value integrals*, Numer. Math. **43** (1984), 161-173.

[9] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **213** (1979).

[10] M.A. Sheshko, *On the convergence of cubature processes for two-dimensional integrals*, Dokl. Akad. Nauk. BSSR **23** (1979), 293-296.

[11] P. Rabinowitz, *Convergence results for piecewise linear quadratures for Cauchy principal value integrals,* Math. Comp. **51** (1988), 741-747.

[12] D.D. Stancu, *The remainder of certain linear formulas in two variables*, SIAM J. Num. Anal. **1** (1964), 137-163.

Received: September 30, 1998.

Prof. Laura Gori
Dipartimento di Metodi e Modelli Matematici
Università “La Sapienza”
Via Scarpa 16, 00161 Roma, Italy
e-mail `gori@dmmm.uniroma1.it`

Prof. Laura Lo Cascio
Dipartimento di Metodi e Modelli Matematici
Università “La Sapienza”
Via Scarpa 16, 00161 Roma, Italy
e-mail `locascio@dmmm.uniroma1.it`

Prof. Elisabetta Santi
Dipartimento di Energetica
Universitá dell’Aquila,
Localitá Monteluco
67040 L’Aquila, Italy
e-mail: `esanti@dsiaq1.ing.univaq.it`