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CONVEX FUNCTIONS OF ORDER n ON UNDIRECTED NETWORKS

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Abstract. In this paper we introduce the convex (nonconcave, polynomial, nonconvex, respective concave) functions of order n on undirected networks. We study some properties of them. Finally we frame these functions in allure theory introduced by E. Popoviciu (1983). We adopt the definition of network as metric space introduced by P. M. Dearing and R. L. Francis (1974).

Key words and phrases: networks, convex functions of order n, allure.

1. PRELIMINARY NOTIONS AND RESULTS

The definition of network as metric space was introduced in [1] and was used in several papers (see, e.g., [2], [4], [3] etc.).

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex we associate a point v_i from an euclidean space X. This yields a finite subset $V = \{v_1, ..., v_m\}$ of X, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called edge of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length l_{ij} and denote by E the set of all edges. We define the network $N = (V, E)$ by

$$N = \{x \in X \mid \exists\, (w_i, w_j) \in A \text{ so that } x \in [v_i, v_j]\}.$$

It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in X. Suppose that for each $[v_i, v_j] \in U$ there exists a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0, 1]$ with $\theta_{ij}(v_i) = 0$, $\theta_{ij}(v_j) = 1$ and $\theta_{ij}([v_i, v_j]) = [0, 1]$.

We denote by T_{ij} the inverse function of θ_{ij}.

Any connected and closed subset of an edge bounded by two points x and y of $[v_i, v_j]$ is called a closed subedge and is denoted by $[x, y]$. If one or

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both of x, y are missing we say than the subedge is open in x, or in y or is open and we denote this by $(x, y]$, $[x, y)$ or (x, y) respectively. Using θ_{ij}, it is possible to compute the length of $[x, y]$ as

$$l([x,y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}.$$

Particularly we have

$$l([v_i, v_j]) = l_{ij},\ l([v_i, x]) = \theta_{ij}(x)\, l_{ij}$$

and

$$l([x, v_j]) = (1 - \theta_{ij}(x))\, l_{ij}.$$

A path $L(x, y)$ linking two points x and y in N is a sequence of edges and at most two subedges at extremities, starting at x and ending at y. If $x = y$ then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by $l(L(x, y))$. If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for any points $x, y \in N$ there exists a path $L(x, y) \subset N$.

A connected network without cycles is called tree.

Let $L^*(x, y)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic. One defines a distance on N as follows:

DEFINITION 1.1. [1] *For any $x, y \in N$, the distance from x to y, $d(x, y)$ in the network N is the length of a shortest path from x to y:*

$$d(x, y) = l(L^*(x, y)).$$

It is obvious that (N, d) is a metric space.

For $x, y \in N$, we denote

$$\langle x, y\rangle = \{z \in N \,|\, d(x, z) + d(z, y) = d(x, y)\}, \tag{1}$$

and $\langle x, y\rangle$ is called the metric segment between x and y.

DEFINITION 1.2. [1] *A set $D \subset N$ is called d-convex if $\langle x, y\rangle \subset D$ for all $x, y \in D$.*

We consider now two points $x, y \in N$, $D(x, y) \subset \langle x, y\rangle$ a shortest path from x to y and a function $f : N \to \mathbb{R}$. We consider also a nonnegative integer $n \geqslant 0$, and the distinct points

$$x_1, \ldots, x_{n+1} \tag{2}$$

such that $\{x_1, \ldots, x_{n+1}\} \subset D(x, y)$.

In [3] E. Iacob denoted:

$$\mathcal{P}_n(x) = \left\{ P : D(x, y) \to \mathbb{R} \,|\, P(t) = \sum_{k=0}^{n} c_k d^k(x, t),\ c_k \in \mathbb{R} \right\}.$$

The elements of $\mathcal{P}_n(x)$ are called metric polynomials. E. Iacob established that there exists a single metric polynomial $P^* \in \mathcal{P}_n(x)$ which is equal with f on the points (2). The polynomial P^* was denoted with $L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f)$ and was called interpolation metric polynomial of Lagrange type.

THEOREM 1.1. [3] *The metric polynomial*

$$L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f) : D(x, y) \to \mathbb{R},$$

$$L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f)(t) =$$

$$= \sum_{i=1}^{n+1} \left(f(x_i) \frac{(d(x,t) - d(x,x_1)) \ldots (d(x,t) - d(x,x_{i-1}))}{(d(x,x_i) - d(x,x_1)) \ldots (d(x,x_i) - d(x,x_{i-1}))} \cdot \frac{(d(x,t) - d(x,x_{i+1})) \ldots (d(x,t) - d(x,x_{n+1}))}{(d(x,x_i) - d(x,x_{i+1})) \ldots (d(x,x_i) - d(x,x_{n+1}))} \right) \tag{3}$$

belongs to the set $\mathcal{P}_n(x)$ and satisfies the conditions

$$L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f)(x_i) = f(x_i),\ \textit{for } i = \overline{1, n+1}. \tag{4}$$

Moreover, $L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f)$ is the unique metric polynomial in $\mathcal{P}_n(x)$ which satisfies the conditions (4).

In [5] the coefficient c_n of $L(\mathcal{P}_n(x); x_1, \ldots, x_{n+1}; f)$ corresponding to $d^n(x, t)$ was denoted by $c_n := [x_1, x_2, \ldots, x_{n+1}; f]$ and was called the divided difference of the function f on the points (2) relative to x. In the sequel we only consider divided differences relative to the fixed point x.

The divided differences have the following properties.

THEOREM 1.2. [5] *For every distinct points $x_1, \ldots, x_{n+1} \in D(x, y)$ we have*

$$[x_1, x_2, \ldots, x_{n+1}; f] =$$

$$= \sum_{i=1}^{n+1} \left(f(x_i) \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \cdot \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+1}))} \right). \tag{5}$$

As in [6] we denote with $\mathcal{D}_{x_1, x_2, \ldots, x_{n+1}}$ the set of all functions defined on the distinct points (2) and we define the functional

$$a_n^{x_1, x_2, \ldots, x_{n+1}} : \mathcal{D}_{x_1, x_2, \ldots, x_{n+1}} \to \mathbb{R}$$

by

$$a_n^{x_1, x_2, \ldots, x_{n+1}}(f) = [x_1, x_2, \ldots, x_{n+1}; f].$$

THEOREM 1.3. [5] *The functional $a_n^{x_1, x_2, \ldots, x_{n+1}}$ is linear, so*

$$a_n^{x_1, x_2, \ldots, x_{n+1}}(f + g) = a_n^{x_1, x_2, \ldots, x_{n+1}}(f) + a_n^{x_1, x_2, \ldots, x_{n+1}}(g),$$

respectively

$$a_n^{x_1, x_2, \ldots, x_{n+1}}(\alpha f) = \alpha a_n^{x_1, x_2, \ldots, x_{n+1}}(f),$$

for all $f, g \in \mathcal{D}_{x_1, x_2, \ldots, x_{n+1}}$ and $\alpha \in \mathbb{R}$.

THEOREM 1.4. [5] *For a point* $x_{n+2} \in D(x,y)$ *different from every point* $x_1, x_2, ..., x_{n+1}$ *we have*

$$\begin{aligned} &f(x_{n+2}) - L(\mathcal{P}_n(x); x_1, x_2, ..., x_{n+1}; f)(x_{n+2}) = \\ &= (d(x, x_{n+2}) - d(x, x_1)) \cdot (d(x, x_{n+2}) - d(x, x_2)) \cdot \\ &\cdots \cdot (d(x, x_{n+2}) - d(x, x_{n+1})) \cdot [x_1, x_2, ..., x_{n+2}; f]. \end{aligned} \tag{6}$$

DEFINITION 1.3. *We say that the distinct points* $x_1, ..., x_n \in D(x,y)$ *form a metric succession relative to* x*, if the following inequalities are satisfied:*

$$d(x, x_1) < d(x, x_2) < ... < d(x, x_n).$$

COROLLARY 1.5. [5] *If the points* $x_1, x_2, ..., x_{n+1}, x_{n+2}$ *form a metric succession relative to* x *then the coefficient*

$$\begin{aligned} &(d(x, x_{n+2}) - d(x, x_1)) \cdot (d(x, x_{n+2}) - d(x, x_2)) \cdot ... \cdot \\ &\cdot (d(x, x_{n+2}) - d(x, x_{n+1})) \end{aligned}$$

of the divided difference $[x_1, x_2, ..., x_{n+2}; f]$ *from the relation (6) is positive. So, in these conditions, the sign of the difference*

$$f(x_{n+2}) - L(\mathcal{P}_n(x); x_1, x_2, ..., x_{n+1}; f)(x_{n+2})$$

depends only on the sign of the divided difference $[x_1, x_2, ..., x_{n+2}; f]$.

2. FUNCTIONS OF ORDER n ON UNDIRECTED NETWORKS

In this section we define the convex (nonconcave, polynomial, nonconvex, respective concave) functions of order n on undirected networks, where $n \geqslant -1$ is an integer number. These are a generalization from undirected networks of real functions of order n introduced in [8] by T. Popoviciu and also studied in [9], [10], [11], [12], [13], [6] etc.

We consider a network N, two points $x, y \in N$, $D(x,y) \subset \langle x, y \rangle$ a shortest path from x to y, a function $f : D(x,y) \to \mathbb{R}$ and an integer number $n \geqslant -1$. We consider also a system of $n+2$ distinct points $x_1, x_2, ..., x_{n+2} \in D(x,y)$. We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

DEFINITION 2.1. *The function* $f : D(x,y) \to \overline{\mathbb{R}}$ *is called convex, (nonconcave, polynomial, nonconvex, respective concave) of order* n *relative to* x *on the distinct points* $x_1, x_2, ..., x_{n+2} \in D(x,y)$ *if the following inequality is satisfied*

$$[x_1, x_2, ..., x_{n+2}; f] > (\geqslant, =, \leqslant, \text{respective } <)\ 0.$$

DEFINITION 2.2. *The function* $f : D(x,y) \to \overline{\mathbb{R}}$ *is called convex (nonconcave, polynomial, nonconvex, respective concave) of order* n *relative to* x *on* $D(x,y)$*, if for any distinct points* $x_1, x_2, ..., x_{n+2} \in D(x,y)$*, the following inequality is satisfied*

$$[x_1, x_2, ..., x_{n+2}; f] > (\geqslant, =, \leqslant, \text{respective } <)\ 0.$$

The function f which have one of the properties of this definition is called also function of order n on $D(x,y)$ relative to x. Using now the theorem 1.4 we obtain the following equivalent definition:

DEFINITION 2.3. *The function* $f : D(x,y) \to \overline{\mathbb{R}}$ *is called convex (nonconcave, polynomial, nonconvex, respective concave) of order* n *relative to* x *on* $D(x,y)$ *if for any distinct points* $x_1, x_2, ..., x_{n+2} \in D(x,y)$ *such that these form a metric succession, the following inequality is satisfied*

$$f(x_{n+2}) - L(\mathcal{P}_n(x); x_1, x_2, ..., x_{n+1}; f)(x_{n+2}) > (\geqslant, =, \leqslant, \text{respective } <)\ 0.$$

In the following we present some basic properties of functions of order n on undirected networks.

THEOREM 2.1. *If* $f : D(x,y) \to \overline{\mathbb{R}}$ *and* $g : D(x,y) \to \overline{\mathbb{R}}$*, are two convex (nonconcave, polynomial, nonconvex, respective concave) functions of order* n *relative to* x *on* $D(x,y)$ *and* α *is a real positive number, then* $f+g$ *and* αf *are also convex (nonconcave, polynomial, nonconvex, respective concave) functions of order* n *relative to* x *on* $D(x,y)$.

Proof. These properties of the functions of order n are a consequence of the properties of the divided differences established in Theorem 1.3. □

THEOREM 2.2. 1. *If* $\{f_k : D(x,y) \to \overline{\mathbb{R}}\}_{k\in\mathbb{N}}$ *is a punctual convergent sequence of convex or nonconcave functions of order* n *relative to* x *on* $D(x,y)$*, then the limit function*

$$f : D(x,y) \to \overline{\mathbb{R}},\ f(z) = \lim_{k\to\infty} f_k(z),$$

is nonconcave of order n *relative to* x *on* $D(x,y)$.

2. *If* $\{f_k : D(x,y) \to \overline{\mathbb{R}}\}_{k\in\mathbb{N}}$ *is a punctual convergent sequence of concave or nonconvex functions of order* n *relative to* x *on* $D(x,y)$*, then the limit function*

$$f : D(x,y) \to \overline{\mathbb{R}},\ f(z) = \lim_{k\to\infty} f_k(z),$$

is nonconvex of order n *relative to* x *on* $D(x,y)$.

3. *If* $\{f_k : D(x,y) \to \overline{\mathbb{R}}\}_{k\in\mathbb{N}}$ *is a punctual convergent sequence of polynomial functions of order* n *relative to* x *on* $D(x,y)$ *then the limit function*

$$f : D(x,y) \to \overline{\mathbb{R}},\ f(z) = \lim_{k\to\infty} f_k(z),$$

is polynomial of order n *relative to* x *on* $D(x,y)$.

Proof. 1. We consider the arbitrary distinct points $x_1, \ldots, x_{n+2} \in D(x,y)$ and the sequence of punctual convergent functions

$$f_k : D(x,y) \to \overline{\mathbb{R}},\ k = 1, 2, \ldots,$$

convex or nonconcave of order n relative to x on $D(x,y)$, having the limit function

$$f : D(x,y) \to \overline{\mathbb{R}},\ f(z) = \lim_{k\to\infty} f_k(z).$$

Then

$$\begin{aligned}
&[x_1, x_2, \ldots, x_{n+2}; f] = \\
&= \sum_{i=1}^{n+2} \left(f(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \right. \\
&\cdot \left. \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+2}))} \right) = \\
&= \sum_{i=1}^{n+2} \left(\lim_{k\to\infty} f_k(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \right. \\
&\cdot \left. \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+2}))} \right) = \\
&= \lim_{k\to\infty} \sum_{i=1}^{n+2} \left(f_k(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \right. \\
&\cdot \left. \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+2}))} \right) = \\
&= \lim_{k\to\infty} [x_1, x_2, \ldots, x_{n+2}; f_k] \geq 0
\end{aligned}$$

since the convexity or nonconcavity of order n of the functions f_k relative to x on $D(x,y)$, imply the nonnegativity of the divided differences $[x_1, \ldots, x_{n+2}; f_k]$ for any $k \in \mathbb{N}$.

The other two affirmations can be proved in a similar way. □

THEOREM 2.3. *If $\mathcal{F}$ is a family of real convex (nonconcave, respective polynomial) functions of order n relative to x on $D(x,y)$, then the function*

$$f_s : D(x,y) \to \overline{\mathbb{R}}, f_s(z) = \sup_{f\in\mathcal{F}} f(z),$$

which is the punctual supremum of this family, is a convex (nonconcave, respective polynomial) function of order n relative to x on $D(x,y)$.

Proof. We consider the family $\mathcal{F}$ of convex (nonconcave, respective polynomial) functions of order n relative to x on $D(x,y)$ and the function

$$f_s : D(x,y) \to \overline{\mathbb{R}},\ f_s(z) = \sup_{f\in\mathcal{F}} f(z),$$

which is the punctual supremum of this family. Then, for arbitrary distinct points $x_1, x_2, \ldots, x_{n+2} \in D(x,y)$, the following relations are satisfied:

$$\begin{aligned}
&[x_1, x_2, \ldots, x_{n+2}; f_s] = \\
&= \sum_{i=1}^{n+2} \left(\sup_{f\in\mathcal{F}} f(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \right. \\
&\cdot \left. \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+2}))} \right) \geq \\
&\geq \sup_{f\in\mathcal{F}} \sum_{i=1}^{n+2} \left(f(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{i-1}))} \right. \\
&\cdot \left. \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot \ldots \cdot (d(x,x_i) - d(x,x_{n+2}))} \right) = \\
&= \sup_{f\in\mathcal{F}} [x_1, x_2, \ldots, x_{n+2}; f].
\end{aligned}$$

Now, since the functions of the family $\mathcal{F}$ are convex (nonconcave, respective polynomial) functions of order n on $D(x,y)$, we have

$$[x_1, x_2, \ldots, x_{n+2}; f] > (\geq, \text{ respective } =)\ 0, \text{ for every } f \in \mathcal{F},$$

so we obtain

$$\sup_{f\in\mathcal{F}} [x_1, x_2, \ldots, x_{n+2}; f] > (\geq, \text{ respective } =)\ 0.$$

This implies that f_s is convex (nonconcave, respective polynomial) of order n on $D(x,y)$. □

We can prove the next theorem in an analogous fashion.

THEOREM 2.4. *If $\mathcal{F}$ is a family of real functions defined on $D(x,y)$, concave (nonconvex, respective polynomial) of order n relative to x on $D(x,y)$, then the function*

$$g : N \to \overline{\mathbb{R}},\ g(x) = \inf_{f\in\mathcal{F}} f(z),$$

which is the punctual infimum of this family, is a concave (nonconvex, respective polynomial) function of order n relative to x on $D(x,y)$.

3. ELEMENTS OF ALLURE THEORY ON UNDIRECTED NETWORKS

In this section we present some elements of allure theory on undirected networks. Even if the term "allure" was used for a very long time in mathematics, an exact definition was given only in 1983 by E. Popoviciu in [7]. We recall the definition in the following.

Consider a set X, a nonempty subset $Y \subset X$, and a partition of the set Y:

$$Y = Y_1 \cup Y_2 \cup ... \cup Y_k$$
$$Y_i \cap Y_j = \varnothing \text{ for } i \neq j$$

Let $\mathcal{U}$ be a set of operators of the form $U : X \to Y$.

DEFINITION 3.1. *The element $x \in X$ is said to have the allure (Y_j, U) if $Ux \in Y_j$.*

DEFINITION 3.2. *The element $x \in X$ is said to have the allure $(Y_j, \mathcal{U})$ if for every $U \in \mathcal{U}$ the element x has the allure (Y_j, U).*

From undirected networks we have immediately the next example of allure.

Example 3.1. (The d-convexity allure of the subsets of an undirected network N.) Let $N = (V, E)$ be an undirected network. Denote $X = \mathcal{P}(N)$, consider $Y = X$, and the partition of Y given by $Y_1 = \varnothing$, $Y_2 = \mathcal{P}(N) \setminus \{\varnothing\}$. Define the operator $U : \mathcal{P}(N) \to \mathcal{P}(N)$ by $U(A) = [A] \setminus A$, for every $A \in \mathcal{P}(N)$, where the set $[A]$ is the union of all closed d-segments determined from each pair of points from A. It is clear that a subset A of N is d-convex if and only if this have the allure $(\{\varnothing\}, U)$.

Example 3.2. The allure of the functions of order n on undirected networks.) Consider a fixed integer number $n \geqslant 0$, a network N, two points $x, y \in N$, a path $D(x, y) \subset \langle x, y \rangle$ and the set $\mathcal{P}_{n+1}(x)$. For every system of $n + 2$ distinct points

$$x_1, x_2, ..., x_{n+2} \tag{7}$$

from $D(x, y)$ we consider the interpolation operator of Lagrange type relative to x

$$L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; \cdot), \tag{8}$$

which attach to a function f, defined on the points (7), the metric polynomial

$$L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; f).$$

Denote by $\mathcal{L}_{n+1}(x)$ the set of operators (8) when is considered all the system of points (7) on $D(x, y)$. We also denote by $\mathcal{P}^+_{n+1}(x)$, $\mathcal{P}^-_{n+1}(x)$, $\mathcal{P}_{n+1}(x)$ the set of polynomials from $\mathcal{P}_{n+1}(x)$ for which the coefficient corresponding of the term $d^{n+1}(x, t)$ is respectively positive, negative or zero. Define the partition for $\mathcal{P}_{n+1}(x)$:

$$\mathcal{P}_{n+1}(x) = \mathcal{P}^-_{n+1}(x) \cup \mathcal{P}_n(x) \cup \mathcal{P}^+_{n+1}(x).$$

Considering now $X = \{f | f : D(x, y) \to \mathbb{R}\}$, $Y = \mathcal{P}_{n+1}(x)$, we have $Y \subset X$. If we apply Definition 3.1 we have that the function f has the allure

$$(\mathcal{P}^+_{n+1}(x), L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; \cdot))$$

if

$$L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; f) \in \mathcal{P}^+_{n+1}(x).$$

In this case f is convex of order n relative to x on the points (7). If f has the allure $(\mathcal{P}^+_{n+1}(x), \mathcal{L}_{n+1}(x))$ then f is convex of order n relative to x on $D(x, y)$. The allure

$$(\mathcal{P}^-_{n+1}(x), L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; \cdot))$$

expresses the concavity of order n relative to x on the points (7), and the allure

$$(\mathcal{P}^-_{n+1}(x), \mathcal{L}_{n+1}(x))$$

expresses the concavity of order n relative to x on $D(x, y)$. The allure

$$(\mathcal{P}_n(x), L(\mathcal{P}_{n+1}(x); x_1, x_2, ..., x_{n+2}; \cdot))$$

expresses the polynomiality of order n relative to x on the points (7), and the allure

$$(\mathcal{P}_n(x), \mathcal{L}_{n+1}(x))$$

expresses the polinomiality of order n relative to x on $D(x, y)$.

Remark 3.1. If we want to determine other types of allures which have the same operator of interpolation (8), we must consider some other partitions. Another immediately partition of $\mathcal{P}_{n+1}(x)$ is

$$\mathcal{P}_{n+1}(x) = [\mathcal{P}^+_{n+1}(x) \cup \mathcal{P}_n(x)] \cup \mathcal{P}^-_{n+1}(x).$$

With this partition the allure

$$(\mathcal{P}^+_{n+1}(x) \cup \mathcal{P}_n(x), L(\mathcal{P}_{n+1}(x); x_1, x_2, ...x_{n+2}; \cdot))$$

expresses the nonconcavity of order n relative to x on the points (7), and the allure

$$(\mathcal{P}^+_{n+1}(x) \cup \mathcal{P}_n(x), \mathcal{L}_{n+1}(x))$$

expresses the nonconcavity of order n relative to x on $D(x, y)$.

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