REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome 29, N° 2, 2000, pp. 161–171

THE UNCENTERED TYPE INCREMENTAL UNKNOWNS FOR A SINGULARLY PERTURBED BILOCAL PROBLEM

A. MIRANVILLE, A. C. MUREŞAN

Abstract. We propose some incremental unknowns which to be adopted for a singularly perturbed boundary value problem.

1. INTRODUCTION

We consider the singularly perturbed boundary value problem:

$$\text{(P1)} \qquad \begin{cases} -\varepsilon u''(x) + a(x)\, u'(x) = f(x), & \text{for } x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where ε is a small positive parameter, $a(x) > 0$ for all $x \in [0,1]$, and the functions a and f are sufficiently smooth. The solution of (P1) has a boundary layer at $x = 1$.

It has long been recognized that difficulties can arise when certain "centered" finite-difference and finite-element methods are applied to (P1) when the diffusion coefficient ε is small. In particular, such schemes when applied to (P1) on a uniform grid have an inherent formal cell Reynolds number limitation. Namely, with a uniform mesh length h and $a(x)$ constant, one finds that the cell Reynolds number $\frac{ah}{\varepsilon}$ must be bounded by some constant depending on the scheme in order to avoid spurious oscillations or gross inaccuracies. For small ε this requires a prohibitive number of grid points and so alternative approaches have been developed. One approach is to use a nonuniform mesh (which must be appropriately chosen) which is very fine "in the boundary layer" and coarser elsewhere. Another approach has been to devise schemes which have no formal cell Reynolds number limitation. Schemes of this type have been constructed by using uncentered ("upwind") differencing for the first derivative term, or, more generally, by adding an "artificial viscosity" to the diffusion coefficient ε, e.g., [10].

2000 AMS Subject Classification: 65L10.

When the discretization is made by finite differences, Temam introduced in [11] the concept of Incremental Unknowns (IU in short). The idea, which stems from dynamical systems approach, consists in writing the approximate solution u_i in the form $u_i = y_i + z_i$, where z is a small increment. Passing from the nodal unknowns u_i to the IUs (y_i, z_i) amounts to a linear change of variables, that is to say, in the language of linear algebra, to the construction of a preconditioner. Many numerical simulations have shown the efficiency of such induced preconditioners.

Numerical solution of a problem such as (P1) using Incremental Unknowns has been considered in [6] and [7] but the IU's that have been used in these articles were connected to the Laplacian only: they were induced only by the discretization matrix of the Laplacian.

In [3], the authors propose a construction of IUs that are more adapted to the problem in the sense that they take into account the convection term in the construction of the IUs, thus leading to the use of an adapted interpolator and of a hierarchichal preconditioner.

We propose in this paper a different approach for the construction of adapted incremental unknowns for (P1): we first make a change of variable (assuming that there exists a grid function, see below) and then discretize the problem. This change of variable allows us to work on a uniform grid, which makes the calculations (in particular via Taylor's expansions) easier; the effects of the grid will then be on the coefficients of the differential operators.

2. THE METHOD

Let $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$, $\exists\, g'(y)$ for all $x \in [0,1]$ and $0 < J(y) := g'(y) < M$, $\forall y \in (0,1)$. Furthermore, in order to obtain a finer resolution near the boundary layer, we assume that $J(1) = 0$.

With the change of variable $x = g(y)$ we obtain from (P1), the following problem:

$$\text{(P2)}\quad \begin{cases} -\varepsilon\left(\dfrac{1}{J(y)}v'(y)\right)' + a(g(y))\,v'(y) = J(y)\,f(g(y)),\ 0<y<1 \\ v(0)=0,\ v(1)=1 \end{cases}$$

Let $\phi(y) = \dfrac{1}{J(y)}v'(y)$. By integration of this equation we obtain:

$$\int_y^{y+h} v'(s)\,ds = \int_y^{y+h} J(s)\,\phi(s)\,ds \text{ and } \int_{y-h}^{y} v'(s)\,ds = \int_{y-h}^{y} J(s)\,\phi(s)\,ds,$$

which can also be written,

$$\text{(1)}\quad \begin{aligned} v(y+h) - v(y) &= \int_y^{y+h} J(s)\,\phi(s)\,ds \text{ and} \\ v(y) - v(y-h) &= \int_{y-h}^{y} J(s)\,\phi(s)\,ds. \end{aligned}$$

Let us consider the Taylor expansion of the function ϕ:

$$\text{(2)}\quad \phi(s) = \phi(y) + (s-y)\,\phi'(y) + \frac{1}{2}(s-y)^2\,\phi''\left(y+\theta^+(s-y)\right), \quad y \le s \le y+h,\ 0<\theta^+<1,$$

$$\text{(3)}\quad \phi(s) = \phi(y) + (s-y)\int_y^{y+h} J(s)\,ds\,\phi'(y) + \frac{1}{2}(s-y)^2\,\phi''\left(y+\theta^-(s-y)\right), \quad y-h \le s \le y,\ 0<\theta^-<1.$$

Injecting these expressions of ϕ in (1), we find from (2) and (3):

$$v(y+h) - v(y) = \phi(y)\int_y^{y+h} J(s)\,ds + \phi'(y)\int_y^{y+h}(s-y)\,J(s)\,ds + \frac{1}{2}\int_y^{y+h}(s-y)^2\,J(s)\,\phi''\left(y+\theta^+(s-y)\right)ds,$$

$$v(y) - v(y-h) = \phi(y)\int_{y-h}^{y} J(s)\,ds + \phi'(y)\int_{y-h}^{y}(s-y)\,J(s)\,ds + \frac{1}{2}\int_{y-h}^{y}(s-y)^2\,J(s)\,\phi''\left(y+\theta^-(s-y)\right)ds.$$

If we denote

$$a^+(y) = \int_y^{y+h} J(s)\,ds,$$

$$a^-(y) = \int_{y-h}^{y} J(s)\,ds,$$

$$b^+(y) = \int_y^{y+h}(s-y)\,J(s)\,ds,$$

$$b^-(y) = \int_{y-h}^{y}(s-y)\,J(s)\,ds,$$

$$I^{+}(y)=\frac{1}{2}\int_{y}^{y+h}(s-y)^{2}\,J(s)\,\phi''\left(y+\theta^{+}(s-y)\right)\mathrm{d}s,$$

$$I^{-}(y)=\frac{1}{2}\int_{y-h}^{y}(s-y)^{2}\,J(s)\,\phi''\left(y+\theta^{-}(s-y)\right)\mathrm{d}s,$$

we obtain

$$a^{+}(y)\,\phi(y)+b^{-}(y)\,\phi'(y)=v(y+h)-v(y)-I^{+}(y)$$

and

$$a^{-}(y)\,\phi(y)+b^{-}(y)\,\phi'(y)=v(y)-v(y-h)-I^{-}(y),$$

which is a system of two equations with two unknowns $\phi(y)$ and $\phi'(y)$ whose determinant will be denoted by $r(y)$.

PROPOSITION 1. *The functions $a^{\pm}$, b^{+} are non-negative functions and the function r is a negative one.*

Since $r<0$ the system has the solution,

$$\phi(y)=\frac{b^{-}(y)}{r(y)}\left[v(y+h)-v(y)\right]-\frac{b^{+}(y)}{r(y)}\left[v(y)-v(y-h)\right]+ +\frac{b^{+}(y)\,I^{-}(y)-b^{-}(y)\,I^{+}(y)}{r(y)},$$

and

$$\phi'(y)=\frac{a^{+}(y)}{r(y)}\left[v(y)-v(y-h)\right]-\frac{a^{-}(y)}{r(y)}\left[v(y+h)-v(y)\right]+ +\frac{a^{-}(y)\,I^{+}(y)-a^{+}(y)\,I^{-}(y)}{r(y)}.$$

We consider the following approximation for $\phi'(y)$ and $\phi(y)$:

$$\phi'(y)=\frac{a^{+}(y)}{r(y)}\left[v(y)-v(y-h)\right]-\frac{a^{-}(y)}{r(y)}\left[v(y+h)-v(y)\right], \tag{4}$$

and

$$\phi(y)=\frac{v(y)-v(y-h)}{a^{-}(y)}, \tag{5}$$

which is a backward type approximation.

If we substitute in (P2), we obtain the approximation:

$$-\varepsilon\left\{\frac{a^{+}(y)}{r(y)}\left[v(y)-v(y-h)\right]-\frac{a^{-}(y)}{r(y)}\left[v(y+h)-v(y)\right]\right\}+ +\frac{a(g(y))\,J(y)}{a^{-}(y)}\left[v(y)-v(y-h)\right]=J(y)\,f(y). \tag{6}$$

Let $h=\dfrac{1}{2N}$, and for $j=0,1,2,\ldots,2N$, $y_j=jh$. For $y=y_j$ in (1) we have for $j=1,\ldots,2N-1$:

$$\left[\frac{a_jJ_j}{a_j^{-}}-\frac{\varepsilon a_j^{+}}{r_j}\right](v_j-v_{j-1})-\frac{\varepsilon a_j^{-}}{r_j}(v_j-v_{j+1})=J_jf_j, \tag{7}$$

where in general $f_j=f(y_j)$.

Since $v(y_j):=u(g(y_j))=u(x_j)$, for $j=0,1,\ldots,2N$, we can write (7):

$$\left(\frac{a_j}{a_j^{-}}-\frac{\varepsilon a_j^{+}}{r_jJ_j}\frac{(x_{j+1}-x_{j-1})}{2}\right)(u_j-u_{j-1})- -\frac{\varepsilon a_j^{-}\,(x_{j+1}-x_{j-1})}{2r_jJ_j}(u_j-u_{j+1})= =\frac{x_{j+1}-x_{j-1}}{2}f_j,\ \text{for } j=1,\ldots,2N-1. \tag{8}$$

Remark 1. *If $g(y):=y$ then we obtain the classical upwind scheme.*

Let $\alpha_j=\dfrac{\varepsilon a_j^{-}\,(x_{j+1}-x_{j-1})}{2r_jJ_j}$ and $\beta_j=\left(\dfrac{a_j}{a_j^{-}}-\dfrac{\varepsilon a_j^{+}}{r_jJ_j}\right)\dfrac{(x_{j+1}-x_{j-1})}{2}$ for $j=1,\ldots,2N-1$. We obtain a finite dimensional linear system which can be written as

$$\beta_j\,(u_j-u_{j-1})+\alpha_j\,(u_j-u_{j+1})=\frac{x_{j+1}-x_{j-1}}{2}f_j,\ \text{for } j=1,\ldots,2N-1. \tag{9}$$

By using trapezoidal rule:

$$\int_{y_j}^{y_{j+1}}g(s)\,\mathrm{d}s\simeq h\frac{g(y_j)+g(y_{j+1})}{2}=h\frac{x_j+x_{j+1}}{2}$$

we have

$$b_j^{+}=\frac{h}{2}(x_{j+1}-x_j);$$

$$b_j^{-}=-\frac{h}{2}(x_j-x_{j-1});$$

so,

$$r_j=-h\,(x_{j+1}-x_j)\,(x_j-x_{j-1}).$$

As usual when an IU method is implemented, two different kinds of unknowns must be distinguished: those associated with the coarse grid components which are on G_c, and whose indices are even and those associated with the complementary points (odd indices) which are on $G_f\backslash G_c$;

∘ • ∘ • ∘ • ∘ .

• : points in G_c, ∘ : points in $G_f\backslash G_c$.

If we write the system at the complementary points, we obtain

$$(\alpha_{2i+1}+\beta_{2i+1})\,u_{2i+1}-\beta_{2i+1}u_{2i}-\alpha_{2i+1}u_{2i+2}=\frac{x_{2i+2}-x_{2i}}{2}f_{2i+1}.$$

Hence, assuming that $\alpha_{2i+1}+\beta_{2i+1}\neq 0$, we have

$$u_{2i+1}=\frac{1}{\alpha_{2i+1}+\beta_{2i+1}}(\alpha_{2i+1}u_{2i+2}+\beta_{2i+1}u_{2i})+\\+\frac{1}{\alpha_{2i+1}+\beta_{2i+1}}\frac{x_{2i+2}-x_{2i}}{2}f_{2i+1}.$$

We note that u_{2i+1} is expressed as the sum of a convex combination of u_{2i} and u_{2i+2}, which is nothing but a bilinear interpolation scheme, and a correction term whose order is connected to the order of the interpolation scheme. If we set

(10) $$z_{2i+1}=u_{2i+1}-\frac{1}{\alpha_{2i+1}+\beta_{2i+1}}(\alpha_{2i+1}u_{2i+2}+\beta_{2i+1}u_{2i}),$$

then the system, at the complementary points, is reduced to

(11) $$z_{2i+1}=\frac{1}{\alpha_{2i+1}+\beta_{2i+1}}\frac{x_{2i+2}-x_{2i}}{2}f_{2i+1},\ i=0,\ ...,\ N-1.$$

so that these values are now explicit. The incremental unknowns for this problem consist of the numbers $y_{2i}=u_{2i}$, $i=0,...,N$, and, at the points $2i+1$, the numbers z_{2i+1}.

At $j=2i$, $i=1,...,N-1$ (9) using (10) and (11) becomes:

(12) $$\frac{\beta_{2i}\beta_{2i-1}}{\alpha_{2i-1}+\beta_{2i-1}}(u_{2i}-u_{2i-2})+\frac{\alpha_{2i}\alpha_{2i+1}}{\alpha_{2i+1}+\beta_{2i+1}}(u_{2i}-u_{2i+2})=\\=\frac{x_{2i+1}-x_{2i-1}}{2}f_{2i}+\frac{\beta_{2i}}{\alpha_{2i-1}+\beta_{2i-1}}\frac{x_{2i}-x_{2i-2}}{2}f_{2i-1}+\\+\frac{\alpha_{2i}}{\alpha_{2i+1}+\beta_{2i+1}}\frac{x_{2i+2}-x_{2i}}{2}f_{2i+1}.$$

Since the recurrence conditions are satisfied, we can obviously repeat recursively the process described above using $d+1$ embedded grids, that is to say using d levels of IUs.

From the point of view of the matriceal framework, this construction can be summarized by the determination of two matrices S and ^{t}T under and upper triangular respectively such that

$${}^t\mathrm{TAS}$$

is bloc diagonal, A being the discretization matrix.

We first consider two grid levels. The discretization matrix A written with the hierarchical ordering (considering first the coarse grisd unknowns

and then the complementary ones) in the form

$$\tilde{\mathrm{A}}=\begin{pmatrix}\Lambda_1 & \mathrm{B}_1\\ \mathrm{B}_2 & \Lambda_2\end{pmatrix},$$

where Λ_i, $i=1,2$ are invertible diagonal matrices.

Construction of S

We want to construct a matrix S of the form:

$$\mathrm{S}=\begin{pmatrix}\mathrm{I} & 0\\ \mathrm{G}_1 & \mathrm{I}\end{pmatrix},$$

and such that AS is upper triangular. We have:

$$\tilde{\mathrm{A}}\mathrm{S}=\begin{pmatrix}\Lambda_1 & \mathrm{B}_1\\ \mathrm{B}_2 & \Lambda_2\end{pmatrix}\begin{pmatrix}\mathrm{I} & 0\\ \mathrm{G}_1 & \mathrm{I}\end{pmatrix}=\begin{pmatrix}\Lambda_1+\mathrm{B}_1\mathrm{G}_1 & \mathrm{B}_1\\ \mathrm{B}_2+\Lambda_2\mathrm{G}_1 & \Lambda_2\end{pmatrix}.$$

Therefore the under-matrix G_1 satisfies

$$\mathrm{G}_1=-\Lambda_2^{-1}\mathrm{B}_2,$$

hence

$$\mathrm{S}=\begin{pmatrix}\mathrm{I} & 0\\ -\Lambda_2^{-1}\mathrm{B}_2 & \mathrm{I}\end{pmatrix}.$$

Construction of T

We now want to construct a matrix ^{t}T of the form:

$${}^t\mathrm{T}=\begin{pmatrix}\mathrm{I} & \mathrm{G}_2\\ 0 & \mathrm{I}\end{pmatrix},$$

and such that ${}^t\mathrm{T}\tilde{\mathrm{A}}\mathrm{S}$ is block diagonal. We have

$${}^t\mathrm{T}\tilde{\mathrm{A}}\mathrm{S}=\begin{pmatrix}\Lambda_1+\mathrm{B}_1\mathrm{G}_1 & \mathrm{B}_1+\mathrm{G}_2\Lambda_2\\ 0 & \Lambda_2\end{pmatrix},$$

and then G_2 must satisfy

$$\mathrm{B}_1+\mathrm{G}_2\Lambda_2=0.$$

Thus

$${}^t\mathrm{T}=\begin{pmatrix}\mathrm{I} & -\mathrm{B}_1\Lambda_2^{-1}\\ 0 & \mathrm{I}\end{pmatrix},$$

and then $\tilde{\mathrm{A}}$ can be written in the form

$$\mathrm{A}={}^t\mathrm{T}\tilde{\mathrm{A}}\mathrm{S}=\begin{pmatrix}\Lambda_1+\mathrm{B}_1\mathrm{G}_1 & 0\\ 0 & \Lambda_2\end{pmatrix}.$$

We note that since the linear system is non-symmetrical, these IUs lead to a non-symmetrical hierarchical preconditioner.

The first diagonal block of $\hat{\mathrm{A}}$ is still tridiagonal and we can and we can repeat recursively the reduction procedure described above by using d levels of IUs.

We now describe two cases.

If:

1) $g'(y_j) \simeq \dfrac{g(y_{j+1}) - g(y_{j-1})}{2h} = \dfrac{x_{j+1} - x_{j-1}}{2h}$ then,

$$\left[\frac{\varepsilon}{x_{j+1} - x_{j-1}} + \frac{a_j(x_{j+1} - x_{j-1})}{2(x_j - x_{j-1})}\right](u_j - u_{j-1}) + \frac{\varepsilon}{x_{j+1} - x_j}(u_j - u_{j+1}) = \frac{x_{j+1} - x_{j-1}}{2} f_j, \tag{8a}$$

and

$$\alpha_j^{(1)} = \frac{\varepsilon}{x_{j+1} - x_j},$$

$$\beta_j^{(1)} = \frac{\varepsilon}{x_j - x_{j-1}} + \frac{a_j(x_{j+1} - x_{j-1})}{2(x_j - x_{j-1})}.$$

Definition 1. *The U1 incremental unknowns are the numbers* $z_{2j+1}^{(1)}$, $j = 0, \dots, N-1$ *defined by*

$$z_{2j+1}^{(1)} = u_{2j+1} - \frac{1}{\alpha_{2j+1}^{(1)} + \beta_{2j+1}^{(1)}}\left(\beta_{2j+1}^{(1)} v_{2j} + \alpha_{2j+1}^{(1)} v_{2j+2}\right).$$

These U1–IUs are Uncentered Incremental Unknowns defined in [3].

2) $g'(y_j) \simeq \dfrac{g(y_{j+1}) - g(y_j)}{h} = \dfrac{x_{j+1} - x_j}{h}$ then,

$$\left[\frac{\varepsilon}{(x_{j+1} - x_j)(x_j - x_{j-1})} + \frac{a_j}{2(x_j - x_{j-1})}\right]\frac{x_{j+1} - x_{j-1}}{2}(u_j - u_{j-1}) + \varepsilon\frac{x_{j+1} - x_{j-1}}{2(x_{j+1} - x_j)}(u_j - u_{j+1}) = \frac{x_{j+1} - x_{j-1}}{2} f_j, \tag{8b}$$

and

$$\alpha_j^{(2)} = \varepsilon\frac{x_{j+1} - x_{j-1}}{2(x_{j+1} - x_j)},$$

$$\beta_j^{(2)} = \left[\frac{\varepsilon}{(x_{j+1} - x_j)(x_j - x_{j-1})} + \frac{a_j}{2(x_j - x_{j-1})}\right]\frac{x_{j+1} - x_{j-1}}{2}.$$

Definition 2. *The U2 incremental unknowns are the numbers* $z_{2j+1}^{(2)}$, $j = 0, \dots, N-1$ *defined by*

$$z_{2j+1}^{(2)} = u_{2j+1} - \frac{1}{\alpha_{2j+1}^{(2)} + \beta_{2j+1}^{(2)}}\left(\beta_{2j+1}^{(2)} u_{2j} + \alpha_{2j+1}^{(2)} u_{2j+2}\right)$$

3. CONCLUSION

Since the discretization points will be more dense in the boundary layer (near $x = 1$) we may assume that:

$$x_{2i+2} - x_{2i+1} \leqslant x_{2i+1} - x_{2i} \text{ for } i = 0, \dots, N-1.$$

For example, if $g(y) = 1 - (1 - y)^{p+1}$, p nonnegative integer, this condition is satisfied.

In this case, assuming that $a(x) > 0$, we have:

$$\left|z_{2i+1}^{(2)}\right| \leqslant \left|z_{2i+1}^{(1)}\right|, \tag{13}$$

hence we can expect that U2–IUs are better (for preconditioning) than U1–IUs, for this type of convection-diffusion problem $(a(x)) > 0$.

For U1–IUs we have the following a priori estimates [3].

Proposition 2. *The U1 incremental unknowns satisfy the following a priori estimates:*

$$\sum_{j=0}^{N-1} z_{2i+1}^2 \leqslant C \cdot \Delta x,$$

$$\sum_{j=0}^{N-1} (y_{2i+2} - y_{2i})^2 \leqslant C \cdot \Delta x,$$

where $\Delta x = \max_{j \in \{0, \dots, 2N-1\}} (x_{j+1} - x_j)$ *and* C *is a constant independent of the mesh.*

Using (13) and this result we can obtain a priori estimates for U3–IUs.

The numerical example.

We consider the following problem:

$$-\varepsilon u''(x) + u'(x) = 1, \text{ for } x \in (0,1)$$
$$u(0) = u(1) = 0,$$

which have the exact solution $u(x) = \dfrac{\exp\left(\dfrac{x}{\varepsilon}\right) - \exp\left(\dfrac{1}{\varepsilon}\right)}{1 - \exp\left(\dfrac{1}{\varepsilon}\right)} + x - 1.$

We make the change of variable by using function $g(y) = 1 - (1 - y)^{p+1}$ and we consider $p = 2$, $h = \dfrac{1}{2N}$ and $\varepsilon = 0,0001$. We have in the following table the spectral condition number of the matrix $\hat{A}^d$ obtained by using d levels of IUs.

Condition number of the matrix $\hat{A}^d$ using U1–IUs respective U2–IUs.

d	U1–IUs	U2–IUs
$d=2$ $(N=3)$	2075.077	834.843
$d=3$ $(N=4)$	1433.189	662.926
$d=4$ $(N=5)$	1004.740	601.321
$d=5$ $(N=6)$	2149.346	1961.929

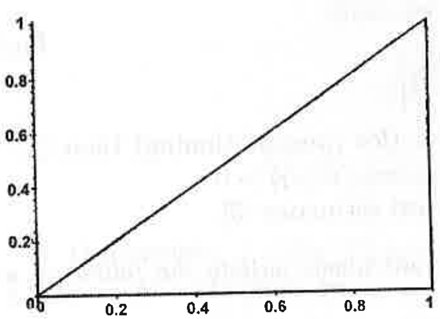

Fig. 1.

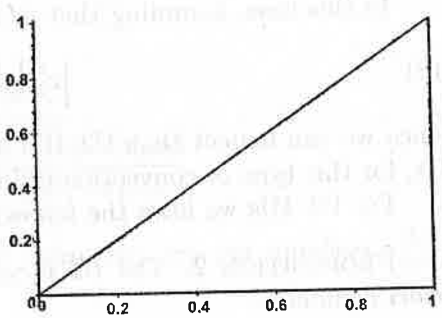

Fig. 2.

In Fig. 1 the exact solution and the approximate solution, in the first case is presented $(N=4)$.

In Fig. 2 the exact solution and the approximate solution, in the second case, is presented $(N=4)$.

We remark that the results obtained by two different methods are very close.

REFERENCES

[1] CHEBAB, J. P., *A Nonlinear Adaptive Multiresolution Method in Finite Differences with Incremental Unknowns*, Modelisation Mathematiques et Analyse Numérique (M^2AN), **29**, no. 4, pp. 451–475, 1995.

[2] CHEBAB, J. P., *Solution of Generalized Stokes Problem Using Hierarchical Methods and Incremental Unknowns*, Appl. Numer. Math., **21**, pp. 9–42, 1996.

[3] CHEBAB, J. P., MIRANVILLE, A., *Incremental Unknowns on Nonuniform Mesches*, Modelisation Mathematiques et Analyse Numerique (M^2AN), **32**, no. 5, pp. 539–577, 1998.

[4] CHEBAB, J. P., MIRANVILLE, A., *Induced Hierarchical Preconditioners: The Finite Difference Case*, Publication ANO-371, 1997.

[5] CHEN, M., TEMAM, R., *Incremental Unknowns for Solving Partial Differential Equations*, Numer. Math., **59**, pp. 255–271, 1991.

[6] CHEN, M., TEMAM, R., *Incremental Unknowns in Finite Differences: Condition Number of the Matrix*, SIAM J. on Matrix Analysis and Applications (SIMAX), **14**, no. 2, pp. 432–455, 1993.

[7] CHEN, M., TEMAM, R., *Incremental Unknowns for Solving Convection-Diffusion Equations*, 1993.

[8] CHEN, M., MIRANVILLE, A., TEMAM, R., *Incremental Unknowns in Finite Differences in Space Dimension 3*, Computational and Applied Mathematics, **14**, no. 3, pp. 219–252, 1995.

[9] GARCIA, S., *Thèse de 1'Université de Paris XI*, Orsay, 1992.

[10] HEMKER, P. W., *A Numerical Study of Stiff Two-Point Boundary Problems, Amsterdam.* 1977.

[11] MUSTĂŢA, C., MUREŞAN, A., MUSTĂŢA, R., *Approximation by Spline Functions of the Solution of a Singularly Perturbed Bilocal Problem*, Revue d'Anal. Numér. et de Théorie de l'Approx., 27, no. 2, pp. 297–308, 1998.

[12] TEMAM, R., Inertial Manifolds and Multigrid Methods, SIAM J. Math. Anal., 21, no. 1, pp. 154–178, 1990.

Received February 22, 2000

Université de Poitiers
Département de Mathématiques
40, avenue du Reatour Pinean
86022 Poitiers Cedex, France
E-mail: miranv@walles.univ-poitiers.fr

"T. Popoviciu" Institute
of Numerical Analysis
P.O. Box 68
3400 Cluj-Napoca, Romania
E-mail: amuresan@ictp-acad.ubbcluj.ro