REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 29, N° 2, 2000, pp. 173–180

SOME REMARKS CONCERNING NORM PRESERVING EXTENSIONS AND BEST APPROXIMATION*

COSTICĂ MUSTĂŢA

Abstract. Let X, Y be two normed spaces, X_1 a subspace of X and $A : X \to Y$ a continuous linear operator. Let us denote $Z_1 = \mathrm{Ker}(A|_{X_1})$, $Z = \mathrm{Ker}A$ and for $x \in X$, $E(x) = \{y \in X : Ax = Ay$ and $\|y\| = \|Ax\| / \|A\|\}$ and $E_1(x) = \{y_1 \in X_1 : Ax = Ay_1$ and $\|y_1\| = \|Ax\| / \|A\|\}$.

One gives the relations between the sets $E(x)$, $E_1(x)$ and $P_Z(x)$, $P_{Z_1}(x)$ where $P_C(x) := \{y \in C : \|x - y\| = d(x, C)\}$. An application is considered.

Let X be a real normed space and M a nonvoid closed subset of X. For $x \in X$ let

$$d(x, M) = \inf\{\|x - y\| : y \in M\}$$

be the *distance* from x to M and let

$$P_M(x) := \{y \in M : \|x - y\| = d(x, M)\}$$

be the set of nearest points from x in the set M.

If $P_M(x) \neq \emptyset$ for every $x \in X$ then the set M is called *proximinal*, if $P_M(x)$ is a singleton for every $x \in X$ then M is called *chebyshevian* and if $P_M(x) = \emptyset$ for every $x \in X \setminus M$ then the set M is called *antiproximinal.*

For a subspace Y of X let

$$Y^\perp = \{x^* \in X^* : x^*|_Y = 0\}$$

be the *annihilator* of the subspace Y in the conjugate space X^* of X.

R.R. Phelps [13] studied the relation between the norm-preserving extension properties of the space Y^* with respect to X^* and the best approximation properties of $Y^\perp$. Namely, he proved that every $y^* \in Y^*$ has a unique norm-preserving extension $x^* \in X^*$ if and only if $Y^\perp$ is a chebyshevian subspace of X^*. By the Hahn-Banach extension theorem, every $y^* \in Y^*$ has at least

2000 AMS Subject Classification: 41A65.
*Supported by the Ministry of Research and Technology (GR 4122/1999).

one norm-preserving extension $x^* \in X^*$. Since then, there have been proved a lot of theorems emphasizing the relations between the extension and best approximation properties for special classes of functions. These results correspond to various extension theorems, such as Tietze extension theorem for continuous functions [6], Mc Shane's extension theorem for Lipschitz functions [7], extension theorems for bilinear functionals on 2-normed spaces [3].

S. Cobzaş [1] proved that all the above mentioned results can be derived from a formula for the distance to the kernel of a continuous linear operator.

For normed spaces X, Y and $A : X \to Y$ a continuous linear operator, let

(1) $$Z = \operatorname{Ker} A = \{x \in X : Ax = 0\}$$

be the kernel of the operator A. Obviously that Z is a closed subspace of X.

For $x \in X$ put

(2) $$E(x) = \left\{ y \in X : Ay = Ax \text{ and } \|y\| = \frac{\|Ax\|}{\|A\|} \right\}.$$

THEOREM 1. S. Cobzaş, [1]. *The following assertions hold:*

1°

(3) $$d(x, Z) \geqslant \frac{\|Ax\|}{\|A\|}$$

2°

(4) $$d(x, Z) = \frac{\|Ax\|}{\|A\|}$$

if and only if there exists a sequence (z_n) in Z such that

(5) $$\|x - z_n\| \to \frac{\|Ax\|}{\|A\|}$$

(a) *If (4) holds then*

(6) $$P_Z(x) = x - E(x)$$

(b) *If there exists $z_0 \in Z$ such that*

(7) $$\|x - z_0\| = \frac{\|Ax\|}{\|A\|}$$

then $z_0 \in P_Z(x)$ and (4) and (6) hold.

By specializing the spaces X, Y and the operator A, S. Cobzaş obtained in the above quoted paper a lot of duality results of Phelps type as well as other results on best approximation.

There are also some duality results as, e.g., those concerning norm-preserving extensions of convex or star-shaped Lipschitz functions (see [2], [9]) which cannot be derived from the theorem mentioned above. The aim of this paper is to prove a slight extension of Theorem 1 such as to cover these extension results, too.

Let X, Y be normed spaces over the same field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and let $A : X \to Y$ be a continuous linear operator. For a subspace X_1 of X let

$$Z = \operatorname{Ker} A \quad \text{and} \quad Z_1 = \operatorname{Ker}\ (A|\ x_1)$$

For $x \in X$ let

(8) $$E(x) = \left\{ y \in X : Ax = Ay \text{ and } \|y\| = \frac{\|Ax\|}{\|A\|} \right\}$$

and

(9) $$E_1(x) = \left\{ y \in X_1 : Ax = Ay \text{ and } \|y\| = \frac{\|Ax\|}{\|A\|} \right\} = X_1 \cap E(x).$$

Obviously, Z_1 is a subspace of Z and $E_1(x) \subseteq E(x)$, for $x \in X_1$.

THEOREM 2. 1°. *For every $x \in X_1$ we have*

(10) $$d(x, Z_1) \geqslant d(x, Z) \geqslant \frac{\|Ax\|}{\|A\|}.$$

2°. *For $x \in X_1$ we have*

(11) $$d(x, Z_1) = d(x, Z) = \frac{\|Ax\|}{\|A\|}$$

if and only if there exists a sequence (z_n) in Z_1 such that

(12) $$\|x - z_n\| \to \frac{\|Ax\|}{\|A\|}.$$

3°. (a) *If the equalities (11) hold then*

(13) $$P_{Z_1}(x) = x - E_1(x)$$

and

(14) $$P_Z(x) = x - E(x).$$

(b) *If there exists $z_0 \in Z_1$ such that*

(15) $$\|x - z_0\| = \frac{\|Ax\|}{\|A\|}$$

then $z_0 \in P_Z(x)$ and the equalities (11), (13) and (14) hold.

Proof. 1°. Let $x \in X_1$. For every $z \in Z_1$ we have

$$\|Ax\| = \|Ax - Az\| = \|A(x-z)\| \leqslant \|A\| \|x-z\|$$

implying

$$\|x-z\| \geqslant \frac{\|Ax\|}{\|A\|}, \text{ for all } z \in Z_1,$$

so that

$$d(x, Z_1) \geqslant \frac{\|Ax\|}{\|A\|}.$$

Similarly

$$d(x, Z) \geqslant \frac{\|Ax\|}{\|A\|}.$$

Since Z_1 is a subspace of Z it follows that

$$d(x, Z_1) \geqslant d(x, Z) \geqslant \frac{\|Ax\|}{\|A\|},$$

for every $x \in X_1$.

2° If $d(x, Z_1) = \frac{\|Ax\|}{\|A\|}$ then, by the definition of $d(x, Z_1)$, there exists a sequence (z_n) in Z_1 such that

$$\|x - z_n\| \to \frac{\|Ax\|}{\|A\|}.$$

Conversely, if (z_n) is a sequence in Z_1 such that $\|x - z_n\| \to \frac{\|Ax\|}{\|A\|}$, then, since $\|x - z_n\| \geqslant d(x, Z_1)$, $n \in \mathbb{N}$, we get

$$d(x, Z_1) \leqslant \lim_{n\to\infty} \|x - z_n\| = \frac{\|Ax\|}{\|A\|},$$

which, combined with (10), gives

$$d(x, Z_1) = \frac{\|Ax\|}{\|A\|}.$$

(a) If $x \in X_1$ is such that $d(x, Z_1) = \frac{\|Ax\|}{\|A\|}$ then the following equivalences hold

$$z \in P_{Z_1}(x) \Longleftrightarrow z \in Z_1 \text{ and } \|x - z\| = d(x, Z_1) = \frac{\|Ax\|}{\|A\|}$$
$$\Longleftrightarrow x - z \in E_1(x) \Longleftrightarrow z \in x - E_1(x).$$

Taking into account (11) one obtains the equivalences

$$z \in P_Z(x) \Longleftrightarrow z \in Z \text{ and } \|x - z\| = d(x, Z) = \frac{\|Ax\|}{\|A\|}$$
$$\Longleftrightarrow x - z \in E(x) \Longleftrightarrow z \in x - E(x).$$

(b) Let $z_0 \in Z_1$ be such that

$$\|x - z_0\| = \frac{\|Ax\|}{\|A\|} = d(x, Z_1) = d(x, Z).$$

It follows that (12) holds for $z_n = z_0$, $n = 1, 2, \ldots$, so that by the point 2°of the theorem, (11), (13) and (14) hold. □

Application

Let X be a real normed space and Y a nonvoid convex subset of X containing 0.

Consider the space

(16) $$Lip_0 Y = \{f : Y \to \mathbb{R} : f \text{ is a Lipschitz on } Y \text{ and } f(0) = 0\}$$

equipped with the *Lipschitz norm*

(17) $$\|f\|_Y = \sup\left\{\frac{|f(y_1) - f(y_2)|}{\|y_1 - y_2\|} y_1, y_2 \in Y,\ y_1 \neq y_2\right\}.$$

The space $Lip_0 X$ and the Lipschitz norm $\|\cdot\|_X$ are defined similarly.

By the theorem of McShane [7], [4], the space $Lip_0 Y$ has the *extension property* with respect to $Lip_0 X$, i.e., for every $f \in Lip_0 Y$ there exists $f \in Lip_0 X$ such that

$$F|_Y = f \text{ and } \|F\|_X = \|f\|_Y.$$

DEFINITION 1. *A function $f \in Lip_0 Y$ is called convex if*

(18) $$f(\alpha y_1 + (1-\alpha) y_2) \leq \alpha f(y_1) + (1-\alpha) f(y_2)$$

for all $y_1, y_2 \in Y$ and all $\alpha \in [0,1]$, and starshaped if

(19) $$f(\alpha y) \leq \alpha f(y)$$

for every $y \in Y$ and every $\alpha \in [0,1]$.

Obviously that every convex function $f \in Lip_0 Y$ is starshaped.

DEFINITION 2. *A subset C of a vector space X is called a convex cone if*
(a) $x + y \in C$ for every $x, y \in C$, and
(b) $\lambda x \in C$ for every $x \in C$ and $\lambda \geq 0$.

Denoting by K_Y (respectively by S_Y) the sets of all convex (respectively starshaped) functions in $Lip_0 Y$, it follows that K_Y and S_Y are convex cones in $Lip_0 Y$.

The sets of convex (starshaped) Lipschitz functions in $Lip_0 X$ are denoted by K_X (respectively by S_X). Again they are convex cones in $Lip_0 X$.

By McShane's theorem ([7], [4]), for every $f \in Lip_0 Y$ the function

(20) $$F(x) = \inf\{f(y) + \|f\|_Y \|x - y\| : y \in Y\},\ x \in X,$$

is a norm preserving extensions of f, i.e.,

(21) $$F|_Y = f \quad \text{and} \quad \|F\|_X = \|f\|_Y$$

a) if $f \in K_Y$ then the function F given by (20) belongs to K_X, i.e., K_Y has the extension property with respect to K_X;

b) if $f \in S_Y$ then $F \in S_X$, so that S_Y has the extension property with respect to S_X, too.

Consider the subspaces

(22) $$X_c = K_X - K_X$$

and

(23) $$X_s = S_X - S_X$$

generated by the cones K_X and S_X, respectively. We have $X_c \subset X_s \subset Lip_0 X$.

Take in Theorem 2, A to be the restriction operator $r : Lip_0 X \to Lip_0 Y$ defined by

(24) $$r(F) = F|_Y,\ F \in Lip_0 X.$$

The operator r is linear, continuous, and, since

$$\|r(F)\|_Y = \|F|_Y\|_Y \leq \|F\|_X,$$

it follows $\|r\| \leq 1$.

For $f \in Lip_0 Y$ put

(25) $$E(f) = \{F \in Lip_0 X : F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\}.$$

By McShane's theorem [7] the set $E(f)$ is nonempty for every $f \in Lip_0 Y$.

If $f \in K_Y$ then

(26) $$E_c(f) = \{F \in E(f) : F \in K_X\} \neq \varnothing,$$

(27) $$E_s(f) = \{F \in E(f) : F \in S_X\} \neq \varnothing,$$

It follows that for $f \in S_Y$

$$E_s(f) \subseteq E(f)$$

and for $f \in K_Y$

$$E_c(f) \subseteq E_s(f) \subseteq E(f).$$

Let

$$Y^{\perp} = \{F \in Lip_0 X : F|_Y = 0\}$$
$$Y_c^{\perp} = Y^{\perp} \cap X_c$$

and

$$Y_s^{\perp} = Y^{\perp} \cap X_s.$$

We have the following result

Proposition 1. *(a) If $F \in S_X$ then $F|_Y \in S_Y$, $E_s(F|_Y) \neq \varnothing$ and $\|r|_{X_s}\| = 1$. Furthermore*

(28) $$d\left(F, Y_s^{\perp}\right) = d\left(F, Y^{\perp}\right) = \|F|_Y\|$$

and

(29) $$F - E_s(F|_Y) = P_{Y_s^{\perp}}(F) \subseteq P_{Y^{\perp}}(F) = F - E(F|_Y)$$

(b) If $F \in K_X$ then $F|_Y \in K_Y$, $E_c(F|_Y) \neq \varnothing$ and $\|r|_{X_c}\| = 1$. Furthermore

(30) $$\begin{aligned} F - E_c(F|_Y) &= P_{Y_c^{\perp}}(F) \subseteq P_{Y_s^{\perp}}(F) \\ &= F - E_s(F|_Y) \subseteq P_{Y^{\perp}}(F) = F - E(f) \end{aligned}$$

Proof. (a) If $F \in E_s(F|_Y)$ then $G = F - \overline{F} \in Y_s^{\perp}$ and

$$\|F - G\|_X = \left\|\overline{F}\right\|_X = \|F|_Y\|_Y = \|r(F - G)\|_Y \leq \|r|_{X_s}\| \|F - G\|_Y$$

implying $\|r|_{X_s}\| = 1$. By Theorem 2.(b), $G \in P_{Y_s^{\perp}}(F)$ so that (28) and (29) hold. The proof of (b) is similar. □

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Received September 21, 1999

"T. Popoviciu" Institute
of Numerical Analysis
C. P. 68,
str. Gh. Bilaşcu 37
3400 Cluj-Napoca, Romania