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VECTORIAL OPTIMIZATION IN LOCALLY CONVEX SPACES ORDERED BY SUPERNORMAL CONES AND EXTENSIONS

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Abstract. This research work was conceived as a completion of [27] with existence results for efficient (Pareto) points in locally convex spaces ordered by supernormal cones, significant comments and recent extensions.

Key words: Supernormal (nuclear) cone, vectorial optimization, efficient (Pareto) point.

1. INTRODUCTION

It is known that the concept of supernormal (nuclear) cone was introduced in Hausdorff locally convex spaces by G. Isac [7] in 1981 and published in 1983 [8]. In every nuclear space [18] a convex cone is supernormal if and only if it is normal (Proposition 6 of [8]) and this is the reason for which such a convex cone was initially called "nuclear cone". The subsequent properties and implications of this notion especially in infinite dimensional Pareto optimization [8], [12], [22–24], [26], the fixed point theory [11], the study of conically bounded sets [1], [2], [9], [10], the geometry of cones [19], the best approximation and optimization in locally convex spaces [13], the vectorial optimization programs with multifunctions and duality [20], [21], Grothendieck's nuclearity [2] and so on show that the nuclear cone is a reinforcement of the normal cone and this fact justifies the definitive name of "supernormal cone". For normed spaces, M.A. Krasnoselski and his colleagues defined the notion of supernormal cone in their important theory concerning with the pointed, closed, convex cones in Banach spaces (see, for instance, [3], [15] and the connected subsequent works). Afterwards, G. Isac [7] extended this concept to its proper framework offered by the separated locally convex spaces and the applications of this extension show that the supernormality has as suitable background the Hausdorff locally convex spaces identically with the nuclearity defined by

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Grothendieck. A study of Grothendieck's nuclearity using supernormal cones is given in [2].

In [27] we find many examples of supernormal cones together with proper remarks (Section 2) and we gave an extension of supernormality to sets accompanied by its immediate connections with a generalization of approximate subdifferential used by J.B. Hiriart–Uruty [6] (Section 3). Here we present some of our existence results for the efficient (Pareto minimum) points which generated their corresponding implications for Pareto maximum points and recent and best extensions.

2. EXISTENCE RESULTS FOR EFFICIENT POINTS IN HAUSDORFF ORDERED LOCALLY CONVEX SPACE

It is known that, in general, to solve a vector optimization problem in an ordered vector space means to find the efficient (Pareto optimums) points of an adequate non-empty set. For this reason, we selected in this section some illustrative existence results of the efficient points in separated locally convex spaces ordered by supernormal cones and by the recent cones defined and studied in [4], [5]. In fact, these cones and a great part of the original and beautiful existence results on the efficient points given in the above mentioned research works were suggested by the existence results of the efficient points obtained through the agency of supernormal cones (see [8], [12], [13], [22], [24], [26] and other conected papers) and by the largest class of convex cones ensuring the existence of the efficient points in compact sets defined in [30].

Let X be a real Hausdorff locally convex space with the topology induced by a family $\mathcal{P} = \{p_\alpha : \alpha \in I\}$ of seminorms ordered by a convex cone K, its topological dual space X^*, A a non-empty subset of X and $\overline{a} \in A$.

DEFINITION 2.1. *We say that $\overline{a}$ is an efficient point (Pareto minimum) for A with respect to K, in notation, $a \in MIN_K(A)$ if $\overline{a}$ satisfies one of the following equivalent conditions*

$$(i)\ A \cap (\overline{a} - K) \subseteq \overline{a} + K; \quad (ii)\ (A + K) \cap (\overline{a} - K) \subseteq \overline{a} + K;$$
$$(iii)\ K \cap (\overline{a} - A) \subseteq -K; \quad (iv)\ K \cap (\overline{a} - A - K) \subseteq -K.$$

We recall that K is *pointed* if $K \cap (-K) = \{0\}$ and *acute* if its closure $\overline{K}$ is pointed. Clearly, $MIN_K(A) = MIN_K(K + A)$ and if K is pointed, then $\overline{a} \in MIN_K(A)$ if and only if $A \cap (\overline{a} - K) = \{\overline{a}\}$, or equivalently, $K \cap (\overline{a} - A) = \{0\}$. In a similar manner one defines the Pareto maximum points.

DEFINITION 2.2. [7] *We say that K is supernormal or nuclear if for every $p \in P$ there exists $f \in X^*$ such that $p(x) \leqslant f(x)$ for all $x \in K$.*

Remark 2.1. *It is clear that every supernormal cone is pointed and the closure of any supernormal cone is also a supernormal cone. If X is a H-Fréchet space, that is, the family PP is countable and every seminorm p_α is generated by a scalar semiproduct $(\cdot,\cdot)_\alpha$, $\alpha \in I$, then the Theorem 2.3 in [26] says that a convex cone K is supernormal in (X, PP) if and only if for every seminorm $p_\alpha \in PP$, there exists $y_\alpha \in X$ such that the subdifferential of p_α at the origin of the space is contained in the translation of the polar cone to K by some linear and continuous functional $(.,y_\alpha)_\beta$, $\beta \in I$.*

The first existence result for the efficient point is based on supernormality of K, the boundedness and completeness of conical (extension) sections induced by non-empty sets and the following important theorem of [8].

THEOREM 2.1. [8] *If K is a supernormal cone in a Hausdorff locally convex space E and S is a non-empty subset of E having the property that there exists a bounded and complete set $S_0 \subseteq S$ with $S \cap (K + x) \subseteq S_0$ for every $x \in S_0$, then there exists $x_0 \in S$ such that $S \cap (K + x_0) = \{x_0\}$.*

THEOREM 2.2. [26] *Let $A \subseteq B \subseteq A + K$. If K is supernormal and $B \cap (A_0 - K)$ is bounded and complete for some non-empty set $A_0 \subseteq A$ then $MIN_K(A) \neq \varnothing$.*

Proof. Let $A' = B \cap (A_0 - K)$ with $A_0 \subseteq A$ such that A' is bounded and complete. Since $A' \cap (a' - K) \subseteq A'$ for every $a' \in A'$, by virtue of Theorem 2.1. it follows that $MIN_K(A') \neq \varnothing$. But $MIN_K(A') \subseteq MIN_K(A)$. Indeed, if $x \in MIN_K(A')$ and we assume that $x \notin A$, then there exist $a \in A$ and $k \in K \setminus \{0\}$, such that $x = a + k$. On the other hand, $x = a_0 - k_1$ with $a_0 \in A_0$ and $k_1 \in K$, therefore $a = x - k = a_0 - (k + k_1)$. Consequently, $a \in A'$ and $x - a \in K \setminus \{0\}$, a contradiction. Hence $MIN_K(A') \subseteq A$.

Suppose now that there exists $x \in MIN_K(A') \setminus MIN_K(A)$. Then, there exists $a_1 \in A$ such that $x - a_1 \in K \setminus \{0\}$. Therefore $a_1 \in x - K \subseteq A_0 - K$ and $a_1 \in A \subseteq B$, that is , $a_1 \in A'$, a contradiction. Consequently, $MIN_K(A') \neq \varnothing$ and $MIN_K(A') \subseteq MIN_K(A)$. This completes the proof. □

Remark 2.2. *The proof of the above theorem shows that if K is supernormal and $A \cap (a - K)$ or $(A + K) \cap (a - K)$ is bounded and complete for some $a \in A$, then $MIN_K(A) \neq \varnothing$. When this boundedness and completeness property holds for every $a \in A$, that is, every section or conical section of A is bounded and complete, then we have the following domination property $A \subseteq MIN_K(A) + K$ which is very useful to establish properties concerning the structure of $MIN_K(A)$.*

COROLLARY 2.2.1 [26] *If A is a non-empty, bounded and closed subset of X and K is well based by a complete set then, $MIN_K(A) \neq \varnothing$ and $A \subseteq MIN_K(A) + K$.*

Proof. Since A is bounded and K is supernormal (Proposition 5 of [8], by Theorem 2.2), it is sufficient to prove that every section of A with respect to K is complete. Let $a \in A$ be an arbitrary element and let $(a_j)_{j\in J}$ be a Cauchy net in $A \cap (a - K)$. Because K is well based by a complete set, there exists a non-empty, convex, bounded and complete set B such that $0 \notin B$ and $K = \bigcup_{\lambda>0} \lambda B$. Hence, for each a_j $(j \in J)$, there exist $\lambda_j > 0$ and $b_j \in B$ with $a_j = a - \lambda_j b_j$. Therefore $(\lambda_j b_j)_{j\in J}$ is a bounded Cauchy net. Since the set B is closed, bounded and $0 \notin B$, there exists a convex and closed neighbourhood V of the zero element in X and $\alpha > 0$ such that $V \cap B = \varnothing$ and $B \subseteq \alpha V$. If p_ν is the Minkowski functional of V, then $1 < p_\nu(b) \leqslant \alpha$ for every $b \in B$ and there exists $M \geqslant 0$ with $\lambda_j \leqslant p_\nu(\lambda_j b_j) \leqslant M$ for all λ_j, that is, $(\lambda_j)_{j\in J}$ is bounded. When $(\lambda_j)_{j\in J}$ contains at least a subnet convergent to zero, then it is clear that a_j tends to a; otherwise, because it is bounded, we can find a subnet $(\lambda_s)_{s\in S}$ convergent to $\lambda_0 > 0$. Since $(a_s)_{s\in S}$ is a Cauchy net, $(b_s)_{s\in S}$ is a Cauchy net in B. Therefore $(b_s)_{s\in S}$ converges to $b_0 \in B$ and $(a_s)_{s\in S}$ is convergent to $a - \lambda_0 b_0$ which implies that $(\lambda_j)_{j\in J}$ converges to $a - \lambda_0 b_0$. So, we have proved the corollary. □

Corollary 2.2.2 [26] *If A is a non-empty, bounded and closed subset of X and K is pointed closed and locally compact, then $MIN_K(A) \neq \varnothing$ and $A \subseteq MIN_K(A) + K$.*

Proof. This follows from the above corollary because in a Hausdorff locally convex space a pointed cone is locally compact if and only if it has a compact generating base. □

Definition 2.3. *A non-empty set $B \subset X$ is K-bounded [20] if there exists a bounded set $B_0 \subset X$ such that $B \subset B_0 + K$ and B is said to be K-closed [16] if the conical extension $B + \overline{K}$ is closed.*

We recall that X is called quasi-complete if every non-empty, bounded and closed subset in X is complete.

Theorem 2.3. [26] *Let X be quasi-complete. If K is a closed and supernormal cone in X, then*

(i) for every non-empty K-bounded and K-closed subset A in X we have $MIN_K(A) \neq \varnothing$ and $A \subseteq MIN_K(A) + K$;

(ii) if the set $B \cap (A_0 - K)$ is K-bounded and K-closed for some non-empty subsets B and A_0 with $A \subseteq B \subseteq A + K$ and $A_0 \subseteq A$, then $MIN_K(A) \neq \varnothing$;

(iii) for every K-bounded and K-closed set $A \subseteq X$, $MIN_K(A) + K = A + K$ and $MIN_K(A)$ is K-bounded and K-closed.

Proof. (i), (ii). In the conditions of the theorem every conical extension section of A is bounded and closed and the results follows by Theorem 2.2 and Remark 2.2. (iii) is based on the inclusion $A \subseteq MIN_K(A) + K$ for every K-bounded and K-closed subset A. □

Before we give an extension of this theorem to ordered Hausdorff topological vector spaces, we recall two basic definitions.

Definition 2.4. [4] *We say that K has property $(*)$ if the set $(M + K) \cap (N - K)$ is bounded whenever M and N are bounded subsets in X.*

Definition 2.5. [4] *We say that K has the property $(**)$ if one of the following equivalent conditions holds:*

(i) any bounded increasing net which is contained in K and in a complete subset of X has a limit;

(ii) any bounded monotone net which is contained in a complete subset of X has a limit.

Remark 2.3. *Every supernormal cone has properties $(*)$ and $(**)$ but there exist convex cones having the properties $(*)$, $(**)$ which are not supernormal. Thus, in the classical Banach spaces $L^p([a,b])$ $(p > 1)$ the usual positive cone is closed, convex, it has the properties $(*)$ and $(**)$ but it is not supernormal. The same conclusion is valid for the cone of nonnegative functions in an Orlicz space (see Example 8 given in Section 2 of [27]).*

Remark 2.4. *Under appropriate conditions, in [5] it is shown that in every separated topological vector space the largest class of convex cones ensuring the existence of the efficient points in any bounded and complete subset coincides with the class of cones having the property $(**)$.*

The announced generalization of Theorem 2.3 is

Theorem 2.4. [4] . *Let X be a Hausdorff topological vector space, $K \subset X$ a convex cone and $A \subset X$ a non-empty set. Suppose that the following conditions hold:*

(i) X is quasi-complete;

(ii) the cone $\overline{K}$ has the properties $(*)$ and $(**)$;

(iii) the set A is K-bounded and K-closed.

Then $MIN_K(A) \neq \varnothing$. If in adition K is correct, that is, $\overline{K} + K \setminus (-K) \subseteq K$, then the domination property $A \subseteq MIN_K(A) + K$ holds.

Following the final remark of Ha T.X.D. in [5] we must mention here that each of the conditions (i)-(iii) in the above theorem cannot be weakened.

Through the agency of the natural extensions with respect to convex cones of upper semicontinuity, boundedness and completeness for multifunctions, the above theorem leads in [4] to obtain a criterion for the existence of the solutions of the vectorial optimization programs

(P) $\mathrm{MIN}_K F(U)$,

where $F : Y \to X$ is a multifunction defined on a topological space Y, U is a non-empty set in Y and X is a topological vector space ordered by a convex cone K.

Let $dom\,(F) = \{y \in Y : F\,(y) \neq \varnothing\}$ be the domain of F. One says that F is upper K-continuous at $y_0 \in dom\,(F)$ if for each neighbourhood V of $F(y_0)$ in X, there exists a neighbourhood W of y_0 in Y such that $F\,(y) \subseteq V + K$, $\forall y \in W \cap dom\,(F)$. If $F(y)$ is K-bounded (K-closed) for every $y \in dom\,(F)$, then F is called K-bounded (K-closed) valued.

Taking into account the Theorem 7.2. in [16], if U is a compact set in Y and $F : Y \to X$ is an upper K-continuous, K-bounded, K-closed valued multifunction, then $F(U)$ is K-bounded and K-closed (this is a generalization to ordered topological vector spaces of the known result which shows that any upper semicontinuous point compact multifunction is compact preserving) we have

THEOREM 2.5. [4] . *If X is quasi-complete, U is a compact set in Y, $F : Y \to X$ is an upper K-continuous, K-bounded and K-closed valued multifunction and the cone $\overline{K}$ has the properties* (∗) *and* (∗∗), *then (P) has solutions.*

Remark 2.5. *When X is a Hausdorff locally convex space ordered by a closed and supernormal cone K, then the Theorem 2.5 is an immediate consequence of Theorem 7.2 in [16] and Theorem 2.3. Since the cones having properties (∗), (∗∗) were conceived as extensions of supernormal cones in order to ensure the existence of efficient points under the same hypotheses on the considered sets, several results of Chapter 4 in [13] concerning the duality for vectorial optimization programs with multifunctions taking values in quasi-complete locally convex spaces ordered by closed and supernormal cones can be extended using such cones as these.*

When $(X, \|.\|)$ is a real linear normed space, P is a closed convex cone in X with its polar cone $P^0 = \{x^* \in X : x^*(x) \leqslant 0,\ \forall x \in P\}$, and S is a closed, convex set in X, then taking into account Theorem 3.1 [29] and Theorem 3.3 [27] we obtain through the agency of usual subdifferential the following characterization of Pareto points.

THEOREM 2.6. *An element $x_0 \in S$ is an efficient point with respect to P if for every $\varepsilon \in (0,1)$, there exists $x^* \in P^0$ such that*

$$\partial\left(\|.\| - x^*\right)(0) \subseteq (s + \varepsilon)\,(x_0 - S)^0\,,\ \forall s \geqslant 0.$$

Clearly, an equivalent form of the above condition can be obtained using Hiriart-Urruty's calculus rules on the subdifferential (see, for example, [6]).

A generalization of this result to separated locally convex spaces will be given in a subsequent research work which we prepare now together with Professor Dr. G. Isac, Royal Military College of St. Jean, Québec, Canada.

3. OPEN PROBLEMS

3.1. If K is a pointed convex cone in a quasi-complete locally convex space such that any K-bounded and K-closed non-empty subset has efficient points with respect to K, then K is supernormal?

3.2 [5] Let E be a quasi-complete Hausdorff topological vector space and K an acute convex cone in E having the property that every non-empty, K-bounded and K-closed subset of E has efficient points. Is it true that K has the properties (∗) and (∗∗)?

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