ON SOARDI'S BERSTEIN OPERATORS OF SECOND KIND

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Abstract. We solve a problem, raised by Paolo Soardi, concerning the shape preserving properties of the Bernstein operators of second kind. We establish also a Voronovskaja-type formula for these operators.

1. INTRODUCTION

Paolo Soardi [6] introduced the following Bernstein operators of second kind: \( \beta_n : C[0,1] \to C[0,1] \)

\[
\beta_n f(x) = \frac{1}{(n+1)2^{n+1}x} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{k} (n+1-2k) \times \]

\[
\times \left( (1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k} (1+x)^k \right) f \left( \frac{n-2k}{n} \right),
\]

where \( n \geq 1, f \in C[0,1], x \in [0,1], \)

They are positive linear operators for which \( \beta_n 1 = 1 \) and \( \beta_n f(1) = f(1), \)

\( n \geq 1, f \in C[0,1], \)

By an intensive use of probabilistic tools Soardi proved

**Theorem 1.1.** Let \( f \) be continuous on \([0,1] \). Then, for all \( n \geq 4, \)

\[
\| \beta_n f - f \| \leq \left( \frac{32}{n} \right) \omega(f, n^{-1/2}),
\]

where \( \| \cdot \| \) is the uniform norm and \( \omega \) is the modulus of continuity.

In the final part of [6] Soardi raised the problem to decide which properties of the function \( f \) are inherited by the \( \beta_n f \)'s.

We shall give some answers to this problem and shall establish a Voronovskaja-type formula for the operators \( \beta_n. \)

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2. PRESERVATION PROPERTIES

The classical Bernstein operators have many well-known preservation properties (see [5], [1]). Here are some similar properties of the operators $\beta_n$.

**Theorem 2.1.** Let $f \in C[0,1]$ and $n \geq 1$.

(i) If $f$ is increasing, then $\beta_n f$ is increasing.

(ii) If $f$ is increasing and convex, then $\beta_n f$ is increasing and convex.

*Proof.* We have for $x \in [0,1]$:

\[ (3) \quad (\beta_n f)'(x) = x^{-2} \sum_{k=0}^{[n/2]-1} p_{n,k}(x) \left( f \left( \frac{n-2k}{n} \right) - f \left( \frac{n-2k-2}{n} \right) \right), \]

where

\[ (4) \quad p_{n,k}(x) = \binom{n}{k} (1-x)^k (1+x)^{n-2k} (n-2k)x - 1 + (1-x)^{n-2k}((n-2k)x + 1). \]

It is easy to prove that $p_{n,k} \geq 0$ on $[0,1]$; thus (i) is proved. Let now $h = [n/2] - 1$. For $0 \leq k \leq h$ let

\[ (5) \quad q_{n,h}(x) := \sum_{j=0}^{h} \binom{n}{j} (n-2j)(1-x)^j (1+x)^{n-2j} - \sum_{j=0}^{h} \binom{n}{j} (n-2j) \left( f \left( \frac{n-2k-2}{n} \right) - f \left( \frac{n-2k}{n} \right) \right) \times \left( (n-2k-1)x^2 - 2x + 2(1-x)^2(1+x)^{n-2j} \right). \]

Then

\[ (6) \quad x^{2n+1}(\beta_n f)'(x) = \sum_{k=0}^{n-1} q_{n,h}(x) - q_{n,h}(-x) \times \left( f \left( \frac{n-2k-2}{n} \right) - 2f \left( \frac{n-2k}{n} \right) - f \left( \frac{n-2k+2}{n} \right) \right) + \sum_{k=0}^{n-1} (q_{n,h}(x) - q_{n,h}(-x)) \left( f \left( \frac{n-2k}{n} \right) - f \left( \frac{n-2k+2}{n} \right) \right). \]

To finish the proof we have only to show that

\[ (7) \quad q_{n,h}(x) \geq q_{n,h}(-x), \quad 0 \leq k \leq h, \quad 0 \leq x \leq 1. \]

This will be accomplished if we prove that the inequality

\[ (8) \quad 2 \left( (1-x)^{2n+1} - (1+x)^{2n+1} \right) \geq \left( n-2l \right) \left( (1-x)^{2n} + (1+x)^{2n} \right) + \left( n-2l+1 \right) \left( (1-x)^{2n+1} - (1+x)^{2n+1} \right) \]

holds for $0 \leq k \leq h$, $0 \leq j \leq n$, $0 \leq x \leq 1$.

Denote $m := n - 2j > 0$, $p := k - j$, $t := \frac{n-2j}{m}$ in [0,1].

Then

\[ (9) \quad m - 2p - 1 = n - 2k - 1 > 0. \]

The inequality (8) is equivalent to

\[ (10) \quad 2(1+t) \left( (1+t)^{2n-1} - m(t^{n-2p-1}) \right) \geq m(m - 2p - 1)(t^{n-2p-1} - t^p)(1-t). \]

Dividing by $(1-t)^3$ we find that (10) is implied by

\[ (11) \quad 2(1+t) \sum_{i=0}^{m-2p-3} (1+t \ldots + t^i)(1+t \ldots + t^{m-2p-3}) \leq m(m - 2p - 1)(1+t \ldots + t^{n-2p-2}). \]

Let $r := m - 2p - 2$; (11) is equivalent to

\[ (12) \quad \sum_{i=0}^{r} (4i(r+i) + 2r)i^i \leq \sum_{i=0}^{r} m(r+1)i^i. \]

The inequality $4i(r+i) + 2r \leq m(r+1)$ (for $0 \leq i \leq r$) is a consequence of (9) and so (12) is true; this finishes the proof.

**Remark 2.2.** Let $c_i(x) = x_i, x \in [0,1]$. Since $\beta_n c_i$ is not a polynomial of degree $\leq 1$, we conclude that $\beta_n$ does not preserve the convexity.

**Remark 2.3.** From (3) and (4) we get

\[ (13) \quad (\beta_n f)'(0) = 0, \quad (\beta_n f)'(1) = \frac{n-1}{n} \left[ -\frac{n+1}{n}; f \right], \]

where $[a,b]$ is the divided difference of $f$ corresponding to the nodes $a$ and $b$. In particular,

\[ (14) \quad (\beta_n c_i)'(0) = 0, \quad (\beta_n c_i)'(1) = \frac{n-1}{n}. \]

For $A > 0$ and $0 < a \leq 1$ let us consider the Lipschitz class

\[ \text{Lip}(A,a) = \{ f \in C[0,1] : |f(x) - f(y)| \leq A|x - y|^a, x, y \in [0,1] \}. \]
THEOREM 2.4. For \( n \geq 1 \) and \( t > 0 \) we have
\[
\beta_n(\text{Lip}(A, \alpha)) \subset \text{Lip}(A^{n-1}, \alpha),
\]
\[
\omega(\beta_n f, t) \leq \frac{2n-1}{n} \omega(f, t).
\]

Proof. In view of [2], Corollaries 6 and 7, we only need to show that
\[
\beta_n(\text{Lip}(A,1)) \subset \text{Lip}(A^{n-1},1).
\]

Let \( f \in \text{Lip}(A,1) \). Then \( A E_1 \pm f \) are increasing and we conclude from
Theorem 2.1 that \( A^p \beta_n E_1 \pm \beta_n f \) are increasing as well. It immediately follows that
\[
f \in \text{Lip}(A^{n-1},1).
\]

From Theorem 2.1 we deduce also that \( \beta_n E_1 \) is increasing and convex, hence \( \beta_n E_1 \) is positive and increasing. It follows that
\[
\|\beta_n E_1 \| = \|\beta_n E_1 (1)\| = \frac{n-1}{n},
\]

hence \( f \in \text{Lip}(A^{n-1},1) \) and the proof is complete. \( \square \)

To conclude this section, let \( n \geq 1 \) and \( m = \lfloor n/2 \rfloor \) be fixed. For \( 0 \leq k \leq m \) and \( 0 \leq x \leq 1 \) set
\[
\psi_{m,k}(x) = \frac{n+1 - 2m + 2k}{n+1} \left( \frac{n+1}{x} - k \right). \\
(1-y)^{n+1} (1+y)^{n+1} - (1-x)^{n+1} (1+x)^{n+1},
\]

Then
\[
\beta_n f(x) = \sum_{k=0}^{m} \beta_{n,k} f(x).
\]

Let \( b_0 < a_1 < \ldots < a_m \); by using the Maclaurin expansion and the
basic composition formula (a generalisation of the Cauchy-Binet formula, see
[4]) we deduce that the system of functions
\[
(\sinh(a_0 t), \sinh(a_1 t), ..., \sinh(a_m t))
\]
is totally positive on \([0, \infty)\); see also [3], p. 161. Now it is easy to infer that the system
\[
(w_{m,0}(x), w_{m,1}(x), ..., w_{m,m}(x))
\]
is totally positive on \([0, 1] \). This fact has the following consequences:

A. If \( \{f_0, f_1, ..., f_p\} \subset C[0,1] \) is totally positive, then \( \beta_n f_0, ..., \beta_n f_p \) is
totally positive.

B. If \( 0 \leq p \leq m \) and \( \{f_0, f_1, ..., f_p\} \subset C[0,1] \) is a Chebyshev system,
then \( \{\beta_n f_0, ..., \beta_n f_p\} \) is a Chebyshev system; moreover, if \( f \in C[0,1] \)
is convex with respect to \( \{f_0, ..., f_p\} \), then \( \beta_n f \) is convex with respect to
\( \{\beta_n f_0, ..., \beta_n f_p\} \).

C. The number of strict sign changes of \( \beta_n f \) is not greater than the
number of strict sign changes in the sequence
\[
(f(\frac{n-2m}{n}), f(\frac{n-2m-2}{n}), ..., f(1)).
\]

(Apply Theorem 3.1 of [3].)

3. MONOTONIC CONVERGENCE

For \( n \geq 1 \) let \( \varphi_n \in C[-1,1] \) be the function determined by the following three conditions:

(a) \( \varphi_n(0) = \left( \frac{n}{2} + 1 \right)^{-1} \); \( \varphi_n(\frac{n-2k}{n}) = \frac{n+1-2k}{n+1-2k}, \ k = 0, 1, \ldots, \left[ \frac{n}{2} \right] \)

(b) \( \varphi_n \) is affine on every interval determined by two consecutive points from

\[
0, \frac{n-2k}{n}, ..., \frac{n-4}{n}, \frac{n-2}{n}, ..., \frac{n}{n}.
\]

(c) \( \varphi_n \) is an even function.

Then \( \varphi_1 = 1, \varphi_n(1) = 1, n = 2, 3, \ldots \), and for \( k = 1, 2, \ldots, \left[ \frac{n}{2} \right], n \geq 2, \)
we have
\[
\varphi_n \left( \frac{n-2k}{n} \right) = \left( 1 - \frac{k}{n} \right)^{-1} \left( \frac{n-1-2k}{n-1-2k} + \frac{k}{n} \right), \ k = 1, 2, \ldots, \left[ \frac{n}{2} \right], n \geq 2.
\]

Using these relations it is easy to construct recursively the graphs of
\( \varphi_2, \varphi_3, \ldots \). Restricted to \([0,1]\), this is a decreasing sequence of increasing,
concave functions.

Consider now the classical Bernstein operators on \([0,1]\):
\[
B_n g = \sum_{k=0}^{n} B_{nk}(g) \left( \frac{n-2k}{n} \right), \quad g \in C[-1,1], \quad t \in [-1,1],
\]
where
\[ b_n,k(t) = 2^{-n} \binom{n}{k} (1 + t)^{n-k} (1 - t)^k, \quad k = 0, 1, \ldots, n. \]

We shall express \( \beta_n f \) by means of \( B_n \) and \( \phi_n \).

Let \( f \in C[0,1] \). Since \( \beta_n f = \beta_n(f - f(0)) + f(0) \), we may suppose that \( f(0) = 0 \). Let \( f_1, f_2 \in C[-1,1] \), \( f_1 \) odd, \( f_2 \) even, \( f_1 + f_2 = f \) on \([0,1]\). Then for \( x \in [0,1] \) we have
\[
\beta_n f(x) = \frac{1 + x}{2} \sum_{k=0}^{n/2} \left( \frac{n - 2k}{n} \right) b_n,k(x) - \frac{1 - x}{2} \sum_{k=0}^{n/2} \left( \frac{n - 2k}{n} \right) b_n,k(-x) =
\]
\[
= \frac{1 + x}{4x} B_n(\phi_n f_1 + f_2)(x) - \frac{1 - x}{4x} B_n(\phi_n f_1 + f_2)(-x).
\]

Since \( \phi_n f_1 \) is odd and \( \phi_n f_2 \) is even, it follows that \( B_n(\phi_n f_1) \) is odd and \( B_n(\phi_n f_2) \) even. We conclude that
\[
(19) \quad \beta_n f(x) = \frac{1}{2} \left( B_n(\phi_n f_1)(x) + \frac{1}{x} B_n(\phi_n f_1)(x) \right)
\]
for all \( n \geq 1, x \in [0,1] \) and \( f \in C[0,1] \) with \( f(0) = 0 \).

**Theorem 3.1.** If \( f \in C[0,1] \) is increasing and convex, then
\[
\beta_n f \geq \beta_{n+1} f \geq f, \quad n = 1, 2, \ldots
\]

**Proof.** Let \( f \in C[0,1], f(0) = 0 \). Then \( f_1(0) = f_2(0) = 0 \). By virtue of (18) we have for \( i = 1, 2 \) and \( k = 0, 1, \ldots, n+1 \),
\[
(\phi_{n+1} f_i) \left( \frac{n + 1 - 2k}{n+1} \right) = (\phi_n f_i) \left( \frac{n + 1 - 2k}{n+1} \right).
\]

This yields \( B_{n+1}(\phi_{n+1} f_i) = B_n(\phi_n f_i) \), \( i = 1, 2 \). Now (19) implies
\[
(22) \quad 2(\beta_n f(x) - \beta_{n+1} f(x)) = B_n(\phi_n f_1)(x) - B_{n+1}(\phi_{n+1} f_1)(x) + \frac{1}{x} B_n(\phi_n f_1)(x) - B_{n+1}(\phi_{n+1} f_1)(x), \quad x \in [0,1].
\]

On the other hand it is well-known that for \( g \in C[-1,1] \) and \( t \in [-1,1] \),
\[
(23) B_n g(t) - B_{n+1} g(t) = \frac{1 - t^2}{n(n+1)2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{n - 2k - 2}{n}, \frac{n - 2k - 1}{n}, \frac{n - 2k}{n} \right) (1 - t)^{n-k-1} \times
\]
\[
\times (1 - t)^k \left[ \frac{n - 2k - 2}{n}, \frac{n - 2k - 1}{n}, \frac{n - 2k}{n} \right].
\]

Let \( x \in [0,1] \); by using (22), (23) and the properties of \( \psi_n \) we find after some calculation:
\[
\beta_{2m} f(x) - \beta_{2m+1} f(x) = \frac{1 - x^2}{x} \sum_{k=0}^{m-1} \binom{2m-k}{k} \left( \frac{2m-1}{2m-k} \left( 1 + x \right)^{2m-k} - \left( 1 + x \right)^{2m-k} \right) \times
\]
\[
\times \left( 2m - 2k - 1 \left[ \frac{m - k - 1}{m}, \frac{2m - 2k - 1}{m} \right] + \frac{m(2m+1)}{(m-k)(2m-k+1)} \left[ \frac{2m - 2k - 1}{m}, \frac{m-k}{m} \right] \right),
\]

\[
(24)
\]

\[
\beta_{2m} f(x) = 2 \left( \frac{1 - x^2}{m} \left[ \frac{2m - 2k - 1}{m - 1} \right] + \frac{m(2m+1)}{(2m-k)(2m-k-1)} \left[ \frac{m-k}{m}, \frac{2m - 2k - 1}{m - 1} \right] \right).
\]

\[
(25)
\]

We have proved (24) and (25) for \( f \in C[0,1], f(0) = 0 \); it is easy to infer that they are valid for an arbitrary \( f \in C[0,1] \).

If \( f \in C[0,1] \) is increasing and convex, then (24) and (25) show that
\( \beta_{2m} f \geq \beta_{2m+1} f, n \geq 1 \). In order to prove the last inequality in (20) we have only to apply Theorem 1.1. \( \square \)

4. A VORONOVSJAKA-TYPE FORMULA

Suppose that \( x \in [-1,1], g \in C[-1,1] \) and \( g'(x) \) is finite; for the classical Bernstein operators \( B_n \) on \([0,1] \), Voronovsja's formula is
\[
\lim_{n \to \infty} n(B_n g(x) - g(x)) = \frac{1 - x^2}{2} g'(x).
\]

\[
(26)
\]
THEOREM 4.1. Let $x \in (0, 1]$ and $f \in C[0,1]$ with $f''(x)$ finite. Then

$$n(b_n f(x) - f(x)) = \frac{1-x^2}{2} f''(x) + \frac{1-x}{x} f'(x).$$

Proof. Consider the functions $\varphi, \psi \in C[-1, 1]$,

$$\varphi(t) = \frac{2|t|}{1 + |t|}, \quad \psi(t) = \frac{2(1 - |t|)}{(1 + |t|)^2}, \quad t \in [-1, 1].$$

It is a matter of calculus to prove that

$$\lim_{n \to \infty} n(\varphi_n - \varphi) = \psi,$$

uniformly on $[-1, 1]$.

Now let $x \in (0, 1]$ and $f \in C[-1, 1]$ with $f''(x)$ finite and $f(0) = 0$. Then

$$n(b_n f(x) - f(x)) = \frac{1}{2} B_n(n(\varphi_n - \varphi) - \psi) f_2(x) +$$

$$+ \frac{1}{2x} B_n(n(\varphi_n - \varphi) - \psi f_2(x)) + \frac{1}{2x} B_n(\psi f_1(x)) +$$

$$+ \frac{1}{2} n(B_n(\psi f_2(x)) - (\psi f_2(x))) + \frac{1}{2x} n(B_n(\psi f_1(x)) - (\psi f_1(x))).$$

By virtue of (28) we have

$$\lim_{n \to \infty} B_n(n(\varphi_n - \varphi) - \psi f_i(x)) = 0, \quad i = 1, 2.$$ Moreover, $B_n(\psi f_i(x)) = \psi(x)f_i(x)$, $i = 1, 2$. From (26) we infer:

$$\lim_{n \to \infty} n(B_n(\psi f_1(x)) - (\psi f_1(x))) = \frac{1-x^2}{2} \psi f_1(x).$$

Summing up, we find

$$\lim_{n \to \infty} n(b_n f(x) - f(x)) = \frac{1-x^2}{2} f''(x) + \frac{1-x}{x} f'(x).$$

Thus we have proved (27) for $f \in C[0,1]$ with $f(0) = 0$ and $f''(x)$ finite; it is easy to infer that (27) is true for an arbitrary $f \in C[0,1]$ with $f''(x)$ finite. This finishes the proof.

REFERENCES


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