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ON SOARDI'S BERSTEIN OPERATORS OF SECOND KIND

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Abstract. We solve a problem, raised by Paolo Soardi, concerning the shape preserving properties of the Bernstein operators of second kind. We establish also a Voronovskaja-type formula for these operators.

1. INTRODUCTION

Paolo Soardi [6] introduced the following Bernstein operators of second kind: $\beta_n : C[0,1] \to C[0,1]$

$$\beta_n f(x) = \frac{1}{(n+1)2^{n+1}x} \sum_{k=0}^{[n/2]} \binom{n+1}{k} (n+1-2k) \times \tag{1}$$

$$\times \left((1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k} (1+x)^k \right) f\left(\frac{n-2k}{n} \right),$$

where $n \geq 1$, $f \in C[0,1]$, $x \in [0,1]$.

They are positive linear operators for which $\beta_n 1 = 1$ and $\beta_n f(1) = f(1)$, $n \geq 1$, $f \in C[0,1]$.

By an intensive use of probabilistic tools Soardi proved

THEOREM 1.1. *Let f be continuous on* [0,1]. *Then, for all $n \geq 4$,*

$$\|\beta_n f - f\| \leq \left(55 + \frac{32}{n} \right) \omega(f, n^{-1/2}), \tag{2}$$

where $\|\cdot\|$ is the uniform norm and ω is the modulus of continuity.

In the final part of [6] Soardi raised the problem to decide which properties of the function f are inherited by the $\beta_n f$'s.

We shall give some answers to this problem and shall establish a Voronovskaja-type formula for the operators β_n.

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2. PRESERVATION PROPERTIES

The classical Bernstein operators have many well-known preservation properties (see [5], [1]). Here are some similar properties of the operators β_n.

THEOREM 2.1. *Let $f \in C[0,1]$ and $n \geq 1$.*
(i) *If f is increasing, then $\beta_n f$ is increasing.*
(ii) *If f is increasing and convex, then $\beta_n f$ is increasing and convex.*

Proof. We have for $x \in [0,1]$

$$(3)\quad (\beta_n f)'(x) = x^{-2}2^{-n-1}\sum_{k=0}^{[n/2]-1} p_{n,k}(x)\left(f\left(\frac{n-2k}{n}\right) - f\left(\frac{n-2k-2}{n}\right)\right),$$

where

$$(4)\quad p_{n,k}^{(x)} = \binom{n}{k}(1-x^2)^k\Big((1+x)^{n-2k}((n-2k)x-1) + (1-x)^{n-2k}((n-2k)x+1)\Big).$$

It is easy to prove that $p_{n,k} \geq 0$ on $[0,1]$; thus (i) is proved. Let now $h = [n/2]-1$. For $0 \leq k \leqslant h$ let

$$(5)\quad q_{n,k}(x) := \sum_{j=0}^{k}\binom{n}{j}\Big((n-2j)(1-x)^k(1+x)^{n-k-1}\cdot((n-2k-1)x^2-2x) + 2(1-x)^j(1+x)^{n-j}\Big).$$

Then

$$(6)\quad x^3 2^{n+1}(\beta_n f)''(x) = \sum_{k=0}^{h-1}(q_{n,k}(x) - q_{n,k}(-x)) \times \left(f\left(\frac{n-2k}{n}\right) - 2f\left(\frac{n-2k-2}{n}\right) + f\left(\frac{n-2k-4}{n}\right)\right) + (q_{n,h}(x) - q_{n,h}(-x))\left(f\left(\frac{n-2h}{n}\right) - f\left(\frac{n-2h-2}{n}\right)\right).$$

To finish the proof we have only to show that

$$(7)\quad q_{n,k}(x) \geq q_{n,k}(-x), \quad 0 \leq k \leq h, \quad 0 \leq x \leq 1.$$

This will be accomplished if we prove that the inequality

$$(8)\quad 2\left((1-x)^j(1+x)^{n-j} - (1+x)^j(1-x)^{n-j}\right) \geq (n-2j)\Big(2x\left((1+x)^k(1-x)^{n-k-1} + (1-x)^k(1+x)^{n-k-1}\right) + (n-2k-1)x^2\left((1+x)^k(1-x)^{n-k-1} - (1-x)^k(1+x)^{n-k-1}\right)\Big)$$

holds for $0 \leq k \leq h$, $0 \leq j \leq k$, $0 \leq x \leq 1$.

Denote $m := n-2j > 0$, $p := k-j$, $t := \frac{1-x}{1+x} \in [0,1]$. Then

$$(9)\quad m-2p-1 = n-2k-1 > 0.$$

The inequality (8) is equivalent to

$$(10)\quad 2(1+t)\left(2(1+t+\ldots+t^{m-1}) - m(t^p + t^{m-p-1})\right) \geq m(m-2p-1)(t^{m-p-1} - t^p)(1-t).$$

Dividing by $(1-t)^2$ we find that (10) is implied by

$$(11)\quad 2(1+t)\sum_{i=0}^{m-2p-3}(1+t+\ldots+t^i)(1+t+\ldots+t^{m-2p-3-i}) \leq m(m-2p-1)(1+t+\ldots+t^{m-2p-2}).$$

Let $r := m-2p-2$; (11) is equivalent to

$$(12)\quad \sum_{i=0}^{r}(4i(r-i)+2r)t^i \leq \sum_{i=0}^{r} m(r+1)t^i.$$

The inequality $4i(r-i)+2r \leq m(r+1)$ (for $0 \leq i \leq r$) is a consequence of (9) and so (12) is true; this finishes the proof. □

Remark 2.2. *Let $e_1(x) = x$, $x \in [0,1]$. Since $\beta_n e_1$ is not a polynomial of degree ≤ 1, we conclude that $\beta_n (n \geq 2)$ does not preserve the convexity.*

Remark 2.3. *From (3) and (4) we get*

$$(13)\quad (\beta_n f)'(0) = 0, \quad (\beta_n f)'(1) = \frac{n-1}{n}\left[\frac{n-2}{n}, 1; f\right],$$

where $[a,b;f]$ is the divided difference of f corresponding to the nodes a and b. In particular,

$$(14)\quad (\beta_n e_1)'(0) = 0, \quad (\beta_n e_1)'(1) = \frac{n-1}{n}.$$

For $A > 0$ and $0 < \alpha \leq 1$ let us consider the Lipschitz class

$$\mathrm{Lip}(A,\alpha) = \{f \in C[0,1] : |f(x) - f(y)| \leq A|x-y|^\alpha, x, y \in [0,1]\}.$$

THEOREM 2.4. *For $n \geq 1$ and $t > 0$ we have*

$$\beta_n(\mathrm{Lip}(A,\alpha)) \subset \mathrm{Lip}(A\left(\frac{n-1}{n}\right)^{\alpha}, \alpha), \tag{15}$$

$$\omega(\beta_n f, t) \leq \frac{2n-1}{n}\omega(f,t). \tag{16}$$

Proof. In view of [2], Corollaries 6 and 7, we only need to show that

$$\beta_n(\mathrm{Lip}(A,1)) \subset \mathrm{Lip}(A\frac{n-1}{n}, 1). \tag{17}$$

Let $f \in \mathrm{Lip}(A,1)$. Then $Ae_1 \pm f$ are increasing and we conclude from Theorem 2.1 that $A\beta_n e_1 \pm \beta_n f$ are increasing as well. It immediately follows that

$$f \in \mathrm{Lip}(A\|(\beta_n e_1)'\|, 1).$$

From Theorem 2.1 we deduce also that $\beta_n e_1$ is increasing and convex, hence $(\beta_n e_1)'$ is positive and increasing. It follows that

$$\|(\beta_n e_1)'\| = (\beta_n e_1)'(1) = \frac{n-1}{n},$$

hence $f \in \mathrm{Lip}\left(A\frac{n-1}{n}, 1\right)$ and the proof is complete. □

To conclude this section, let $n \geq 1$ and $m = [n/2]$ be fixed. For $0 \leq k \leq m$ and $0 \leq x \leq 1$ set

$$w_{n,k}(x) = \frac{n+1-2m+2k}{(n+1)2^{n+1}x}\binom{n+1}{m-k} \cdot$$
$$\cdot\left((1-x)^{m-k}(1+x)^{n+1-m+k} - (1-x)^{n+1-m+k}(1+x)^{m-k}\right).$$

Then

$$\beta_n f(x) = \sum_{k=0}^{m} f\left(\frac{n-2m+2k}{n}\right) w_{n,k}(x).$$

Let $0 < a_0 < a_1 < \ldots < a_m$; by using the Maclaurin expansion and the basic composition formula (a generalisation of the Cauchy-Binet formula, see [4]) we deduce that the system of functions

$$(\sinh(a_0 t), \sinh(a_1 t), \ldots, \sinh(a_n t))$$

is totally positive on $[0,\infty)$; see also [3], p. 161. Now it is easy to infer that the system

$$(w_{n,0}(x), w_{n,1}(x), \ldots, w_{n,m}(x))$$

is totally positive on $[0,1]$. This fact has the following consequences:

A. If $\{f_0, f_1, \ldots, f_p\} \subset C[0,1]$ is totally positive, then $\{\beta_n f_0, \ldots, \beta_n f_p\}$ is totally positive.

B. If $0 \leq p \leq m$ and $\{f_0, f_1, \ldots, f_p\} \subset C[0,1]$ is a Chebyshev system, then $\{\beta_n f_0, \ldots, \beta_n f_p\}$ is a Chebyshev system; moreover, if $f \in C[0,1]$ is convex with respect to $\{f_0, \ldots, f_p\}$, then $\beta_n f$ is convex with respect to $\{\beta_n f_0, \ldots, \beta_n f_p\}$.

C. The number of strict sign changes of $\beta_n f$ is not greater than the number of strict sign changes in the sequence

$$\left(f\left(\frac{n-2m}{n}\right), f\left(\frac{n-2m-2}{n}\right), \ldots, f(1)\right).$$

(Apply Theorem 3.1 of [3].)

3. MONOTONIC CONVERGENCE

For $n \geq 1$ let $\varphi_n \in C[-1,1]$ be the function determined by the following three conditions:

(a) $\varphi_n(0) = \left(\left[\frac{n}{2}\right]+1\right)^{-1}$; $\quad \varphi_n\left(\frac{n-2k}{n}\right) = \frac{n+1-2k}{n+1-k}$, $\quad k = 0, 1, \ldots, \left[\frac{n}{2}\right]$.

(b) φ_n is affine on every interval determined by two consecutive points from the sequence

$$0, \frac{n-2[n/2]}{n}, \ldots, \frac{n-4}{n}, \frac{n-2}{n}, 1.$$

(c) φ_n is an even function.

Then $\varphi_1 = 1$, $\varphi_n(1) = 1$, $n = 2, 3, \ldots$ and for $k = 1, 2, \ldots, \left[\frac{n-1}{n}\right]$, $n \geq 2$, we have

$$\left\{\begin{array}{l} \dfrac{n-2k}{n} = \left(1-\dfrac{k}{n}\right)\dfrac{n-1-2k}{n-1} + \dfrac{k}{n}\dfrac{n-1-2(k-1)}{n-1} \\ \varphi_n\left(\dfrac{n-2k}{n}\right) = \left(1-\dfrac{k}{n}\right)\varphi_{n-1}\left(\dfrac{n-1-2k}{n-1}\right) + \\ \qquad + \dfrac{k}{n}\varphi_{n-1}\left(\dfrac{n-1-2(k-1)}{n-1}\right) \end{array}\right. \tag{18}$$

Using these relations it is easy to construct recursively the graphs of $\varphi_2, \varphi_3, \ldots$. Restricted to $[0,1]$, this is a decreasing sequence of increasing concave functions.

Consider now the classical Bernstein operators on $C[-1,1]$:

$$B_n g(t) = \sum_{k=0}^{n} b_{n,k}(t) g\left(\frac{n-2k}{n}\right), \quad g \in C[-1,1], \quad t \in [-1,1],$$

where

$$b_{n,k}(t) = 2^{-n}\binom{n}{k}(1+t)^{n-k}(1-t)^k, \quad k = 0, 1, \ldots, n.$$

We shall express $\beta_n f$ by means of B_n and φ_n.

Let $f \in C[0,1]$. Since $\beta_n f = \beta_n(f - f(0)) + f(0)$, we may suppose that $f(0) = 0$. Let $f_1, f_2 \in C[-1,1]$, f_1 odd, f_2 even, $f_1 = f_2 = f$ on [0,1]. Then for $x \in [0,1]$ we have

$$\beta_n f(x) = \frac{1+x}{2x}\sum_{k=0}^{[n/2]}(\varphi_n f)\left(\frac{n-2k}{n}\right) b_{n,k}(x) -$$

$$-\frac{1-x}{2x}\sum_{k=0}^{[n/2]}(\varphi_n f)\left(\frac{n-2k}{n}\right) b_{n,k}(-x) =$$

$$= \frac{1+x}{4x}B_n\varphi_n(f_1+f_2)(x) - \frac{1-x}{4x}B_n\varphi_n(f_1+f_2)(-x).$$

Since $\varphi_n f_1$ is odd and $\varphi_n f_2$ is even, it follows that $B_n(\varphi_n f_1)$ is odd and $B_n(\varphi_n f_2)$ even. We conclude that

(19) $$\beta_n f(x) = \frac{1}{2}\left(B_n(\varphi_n f_2)(x) + \frac{1}{x}B_n(\varphi_n f_1)(x)\right)$$

for all $n \geq 1, x \in [0,1]$ and $f \in C[0,1]$ with $f(0) = 0$.

THEOREM 3.1. *If $f \in C[0,1]$ is increasing and convex, then*

(20) $$\beta_n f \geq \beta_{n+1} f \geq f, \quad n = 1, 2, \ldots.$$

Proof. Let $f \in C[0,1], f(0) = 0$. Then $f_1(0) = f_2(0) = 0$. By virtue of (18) we have for $i = 1,2$ and $k = 0, 1, \ldots, n+1$,

(21) $$(\varphi_{n+1} f_i)\left(\frac{n+1-2k}{n+1}\right) = (\varphi_n f_i)\left(\frac{n+1-2k}{n+1}\right).$$

This yields $B_{n+1}(\varphi_{n+1} f_i) = B_n(\varphi_n f_i)$, $i = 1,2$. Now (19) implies

(22) $$2(\beta_n f(x) - \beta_{n+1} f(x)) = B_n(\varphi_n f_2)(x) - B_{n+1}(\varphi_n f_2)(x) + $$
$$+\frac{1}{x}(B_n(\varphi_n f_1)(x) - B_{n+1}(\varphi_n f_1)(x)), \quad x \in [0,1].$$

On the other hand it is well-known that for $g \in C[-1,1]$ and $t \in [-1,1]$,

(23) $$B_n g(t) - B_{n+1} g(t) = \frac{1-t^2}{n(n+1)2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(1+t)^{n-1-k} \times$$
$$\times(1-t)^k\left[\frac{n-2k-2}{n}, \frac{n-2k-1}{n}, \frac{n-2k}{n}; g\right].$$

Let $x \in [0,1]$; by using (22), (23) and the properties of φ_n we find after some calculation:

(24) $$\beta_{2m} f(x) - \beta_{2m+1} f(x) =$$
$$\frac{1-x^2}{xm(2m+1)2^{2m+1}}\sum_{k=0}^{m-1}\binom{2m-1}{k}\big((1+x)^{2m-k}(1-x)^k -$$
$$-(1+x)^k(1-x)^{2m-k}\big) \times$$
$$\times\left(\frac{2m-2k-1}{2m-k}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right] +\right.$$
$$\left.+\frac{m(2m+1)}{(2m-k)(2m-k+1)}\left[\frac{2m-2k-1}{2m+1}, \frac{m-k}{m}; f\right]\right),$$

(25) $$\beta_{2m-1} f(x) - \beta_{2m} f(x) =$$
$$2\left(\frac{1-x^2}{4}\right)^m\binom{2m-2}{m-1}\frac{2m-1}{m(m+1)}\left(f\left(\frac{1}{2m-1}\right) - f(0)\right) +$$
$$+\frac{1-x^2}{xm(2m-1)4^m}\sum_{k=0}^{m-2}\binom{2m-2}{k}$$
$$\big((1+x)^{2m-1-k}(1-x)^k - (1+x)^k(1-x)^{2m-1-k}\big) \times$$
$$\times\left(\frac{2m-2k-2}{2m-k-1}\left[\frac{2m-2k-3}{2m-1}, \frac{m-k-1}{m}, \frac{2m-2k-1}{2m-1}; f\right] +\right.$$
$$\left.+\frac{m(2m-1)}{(2m-k)(2m-k-1)}\left[\frac{m-k-1}{m}, \frac{2m-2k-1}{2m-1}; f\right]\right).$$

We have proved (24) and (25) for $f \in C[0,1]$ with $f(0) = 0$; it is easy to infer that they are valid for an arbitrary $f \in C[0,1]$.

If $f \in C[0,1]$ is increasing and convex, then (24) and (25) show that $\beta_n f \geq \beta_{n+1} f, n \geq 1$. In order to prove the last inequality in (20) we have only to apply Theorem 1.1. □

4. A VORONOVSKAJA-TYPE FORMULA

Suppose that $x \in [-1,1], g \in C[-1,1]$ and $g''(x)$ is finite; for the classical Bernstein operators B_n on $C[-1,1]$, Voronovskaja's formula is

(26) $$\lim_{n\to\infty} n(B_n g(x) - g(x)) = \frac{1-x^2}{2}g''(x).$$

THEOREM 4.1. *Let $x \in (0,1]$ and $f \in C[0,1]$ with $f''(x)$ finite. Then*

$$n(\beta_n f(x) - f(x)) = \frac{1-x^2}{2} f''(x) + \frac{1-x}{x} f'(x). \tag{27}$$

Proof. Consider the functions $\varphi, \psi \in C[-1,1]$,

$$\varphi(t) = \frac{2|t|}{1+|t|}, \quad \psi(t) = \frac{2(1-|t|)}{(1+|t|)^2}, \quad t \in [-1,1].$$

It is a matter of calculus to prove that

$$\lim_{n\to\infty} n\,(\varphi_n - \varphi) = \psi, \text{ uniformly on } [-1,1]. \tag{28}$$

Now let $x \in (0,1]$ and $f \in C[0,1]$ with $f''(x)$ finite and $f(0) = 0$. Then

$$n(\beta_n f(x) - f(x)) = \frac{1}{2} B_n(n(\varphi_n - \varphi) - \psi) f_2(x) \;+$$

$$+\frac{1}{2x} B_n(n(\varphi_n - \varphi) - \psi) f_1(x) + \frac{1}{2} B_n(\psi f_2)(x) + \frac{1}{2x} B_n(\psi f_1)(x) \;+$$

$$+\frac{1}{2} n(B_n(\varphi f_2)(x) - (\varphi f_2)(x)) + \frac{1}{2x} n(B_n(\varphi f_1)(x) - (\varphi f_1)(x)).$$

By virtue of (28) we have $\lim_{n\to\infty} B_n((n(\varphi_n - \varphi) - \psi) f_i)(x) = 0$, $i = 1,2$. Moreover, $\lim_{n\to\infty} B_n(\psi f_i)(x) = \psi(x) f(x)$, $i = 1,2$. From (26) we infer:

$$\lim_{n\to\infty} n(B_n(\varphi f_i)(x) - (\varphi f_i)(x)) = \frac{1-x^2}{2} (\varphi f)''(x), \quad i = 1,2.$$

Summing up, we find

$$\lim_{n\to\infty} n(\beta_n f(x) - f(x)) = \frac{1-x^2}{2} f''(x) + \frac{1-x}{x} f'(x)$$

Thus we have proved (27) for $f \in C[0,1]$ with $f(0) = 0$ and $f''(x)$ finite; it is easy to infer that (27) is true for an arbitrary $f \in C[0,1]$ with $f''(x)$ finite. This finishes the proof. □

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