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ACCELERATING THE CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS*

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Abstract. In a previous paper of us, we have shown that no q-superlinear convergence to a fixed point x^* of a nonlinear mapping G may be attained by the successive approximations when $G'(x^*)$ has no eigenvalue equal to 0. However, high convergence orders may theoretically be attained if one considers perturbed successive approximations.

We characterize here the correction terms which must be added at each step in order to obtain convergence with q-order 2 of the resulted iterates.

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1. INTRODUCTION

We are interested in the convergence of the successive approximations

$$x_{k+1} = G(x_k), \quad k = 0, 1, \ldots \tag{1}$$

to a fixed point $x^* \in \mathrm{int}(D)$ of the nonlinear mapping $G : D \subseteq \mathbb{R}^n \to D$. A classical result on the local convergence of these sequences is given by the Ostrowski theorem. First we remind the definitions of the convergence orders.

Let $\|\cdot\|$ denote a given norm on $\mathbb{R}^n$.

Definition 1. [19, ch. 9] *Let $(x_k)_{k\geq 0} \subset \mathbb{R}^n$ be an arbitrary sequence converging to some $x^* \in \mathbb{R}^n$. The quotient and the root convergence factors are defined for each $\alpha \in [1, +\infty)$ as*

$$Q_\alpha\{x_k\} = \begin{cases} 0, & \textit{if } x_k = x^*, \textit{ for all but finitely many } k, \\ \limsup\limits_{k\to\infty} \dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^\alpha}, & \textit{if } x_k \neq x^*, \textit{ for all but finitely many } k, \\ +\infty, & \textit{otherwise,} \end{cases}$$

$$R_\alpha\{x_k\} = \begin{cases} \limsup\limits_{k\to\infty} \|x_k - x^*\|^{1/\alpha^k}, & \textit{when } \alpha > 1, \\ \limsup\limits_{k\to\infty} \|x_k - x^*\|^{1/k}, & \textit{when } \alpha = 1. \end{cases}$$

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The q- and r-convergence orders are defined by

$$
\begin{aligned}
O_Q\{x_k\} &= \begin{cases} +\infty, & \textit{if} \quad Q_\alpha\{x_k\} = 0, \quad \forall\alpha \in [1, +\infty), \\ \inf\{\alpha \in [1, +\infty) : Q_\alpha\{x_k\} = +\infty\}, & \textit{otherwise} \end{cases} \\
O_R\{x_k\} &= \begin{cases} +\infty, & \textit{if} \quad R_\alpha\{x_k\} = 0, \quad \forall\alpha \in [1, +\infty), \\ \inf\{\alpha \in [1, +\infty) : R_\alpha\{x_k\} = 1\}, & \textit{otherwise.} \end{cases}
\end{aligned}
$$

When $Q_1\{x_k\} = 0$ or $R_1\{x_k\} = 0$, the sequence converges q-, resp. r-superlinearly; if $Q_{\alpha_0}\{x_k\} < +\infty$ for some $\alpha_0 > 1$, one may write

$$
\|x_{k+1} - x^*\| = \mathcal{O}(\|x_k - x^*\|^{\alpha_0}), \quad \text{as } k \to \infty.
$$

The q-convergence rates require conditions stronger than for the r-convergence rates: the q-convergence with a certain order implies r-convergence with at least the same order, the converse being false. We refer the reader to [19, ch. 9] and [23] (see also [25, ch. 3] and [24]) for other different relating results.

The fixed point x^* is an attraction fixed point if there exists an open ball with center at x^* such that for any initial approximation x_0 from that ball, the sequence (1) converges to x^*. We shall denote by $\mathcal{S}$ the set of all such sequences.

The q- and r-factors of the iterative process $\mathcal{S}$ are then defined as

$$
\begin{aligned}
Q_\alpha(\mathcal{S}) &= \sup\{Q_\alpha\{x_k\} : (x_k)_{k\geq 0} \in \mathcal{S}\}, \qquad \text{respectively} \\
R_\alpha(\mathcal{S}) &= \sup\{R_\alpha\{x_k\} : (x_k)_{k\geq 0} \in \mathcal{S}\},
\end{aligned}
$$

the convergence orders being similarly defined as for a single sequence.

Now we can state the following classical result (see also [25, Th. 3.5]).

Theorem 1 (Ostrowski). [20, Th. 22.1], [19, Thms. 10.1.3, 10.1.4] *Assume that the mapping G is differentiable at the fixed point $x^* \in \operatorname{int}(D)$. If the spectral radius of $G'(x^*)$ satisfies*

$$\sigma = \rho(G'(x^*)) < 1,$$

then x^* is an attraction fixed point. Moreover, $R_1(\mathcal{S}) = \sigma$, and if $\sigma > 0$ then $O_R(\mathcal{S}) = O_Q(\mathcal{S}) = 1$.

According to this theorem, when $\sigma = 0$, all the sequences from $\mathcal{S}$ converge r-superlinearly; however, this does not imply that they also converge q-superlinearly (an example is given in [19, E10.1-6]; see also [25, p. 30]). A sufficient condition for q-superlinear convergence of $\mathcal{S}$ is that $G'(x^*) = 0$ [19, Th. 10.1.6] (see also [25, p. 30]). The q-convergence order of $\mathcal{S}$ is even higher under some supplementary smoothness conditions: if G is continuously differentiable on an open neighborhood of the fixed point $x^* \in \operatorname{int}(D)$, G is twice differentiable at x^* and $G'(x^*) = 0$, then $O_R(\mathcal{S}) \geq O_Q(\mathcal{S}) \geq 2$ [19, Th. 10.1.7] (see also [25, Th. 3.6]).

These sufficient conditions (which are not also necessary) ensure the high convergence orders of all the successive approximations near the fixed point, but the restrictions on G' are rather strong. In our paper [8] we have characterized the high convergence orders of only one sequence converging to x^*. We shall consider here only the q-order 2.

Theorem 2. [8] *Assume that the mapping G is differentiable on an open neighborhood of the fixed point x^*, with G' Lipschitz continuous at x^*, i.e. there exists $L, \varepsilon > 0$ such that*

$$\|G'(x) - G'(x^*)\| \leq L\|x - x^*\|, \quad \textit{when } \|x - x^*\| < \varepsilon.$$

Suppose also that $\sigma = \rho(G'(x^*)) < 1$. Let $x_0 \in D$ be an initial approximation such that the sequence of successive approximations converges to x^*. Then $(x_k)_{k\geq 0}$ converges with q-order 2 iff

$$\|G'(x_k)(x_k - G(x_k))\| = \mathcal{O}(\|x_k - G(x_k)\|^2), \quad \textit{as } k \to \infty. \tag{2}$$

This result allowed us to show that condition (2) is in fact equivalent to

$$G'(x^*)(x_{k+1} - x_k) = 0, \quad \forall k \geq k_0,$$

i.e., from a certain step, the corrections $x_{k+1} - x_k$ are eigenvectors corresponding to the eigenvalue 0 of $G'(x^*)$. As a direct consequence, it followed that the trajectories with high convergence orders are restricted to affine subspaces.

Apart from the instability in the presence of errors implied by this result, it also means bad news when $G'(x^*)$ is invertible (i.e. all its eigenvalues are nonzero), since no trajectory may attain high convergence rates.

We are interested in accelerating the convergence of the successive approximations in the case $0 < \sigma < 1$ (or even when $G'(x^*)$ is nonsingular) by adding some correction terms $\delta_k \in \mathbb{R}^n$, i.e., by considering the sequence

$$x_{k+1} = G(x_k) + \delta_k, \quad k = 0, 1, \dots \tag{3}$$

In [8] we have characterized the high convergence orders of this sequence, but δ_k were viewed as error terms, and the above sequence was called as perturbed successive approximations. We shall present a new result which allows (at least theoretically) the computation of some terms δ_k leading to q-quadratic convergence of the iterations (3).

2. ACCELERATED CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS

We have obtained the following result:

THEOREM 3. [8] *Suppose that G satisfies the assumptions of Theorem 2, and that the sequence (3) of perturbed successive approximations converges to x^*. Then the convergence is with q-order 2 iff*

$$\|G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k\| = \mathcal{O}(\|x_k - G(x_k)\|^2), \text{ as } k \to \infty. \tag{4}$$

We note that this result no longer requires $G'(x^*)$ to be singular.

We obtain an equivalent form of condition (4) in the following result:

THEOREM 4. *Suppose that G satisfies the assumptions of Theorem 2, that the sequence $(x_k)_{k\geq 0}$ given by (3) converges to x^*, and that the matrices $I - G'(x_k)$ are invertible starting from a certain step. Then the convergence is with q-order 2 iff*

$$\delta_k = (I - G'(x_k))^{-1}(G'(x_k)(G(x_k) - x_k) + \gamma_k),$$

where $(\gamma_k)_{k\geq 0} \subset \mathbb{R}^n$ is an arbitrary sequence converging to zero with $\gamma_k = \mathcal{O}(\|x_k - G(x_k)\|^2)$, as $k \to \infty$.

Proof. We can easily obtain this result from the previous theorem by denoting

$$\gamma_k = G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k$$

and then computing δ_k. □

Under the assumption that $\|G'(x)\| < q < 1$ for all x in a certain neighborhood of x^*, a natural choice (implied by the Banach lemma) for δ_k is

$$\delta_k = (I + \cdots + G'(x_k)^{i_k})G'(x_k)(G(x_k) - x_k),$$

with i_k such that

$$\frac{q^{i_k+2}}{1-q} \leq K\|x_k - G(x_k)\|$$

for some fixed $K > 0$.

This choice may be useful when the powers of $G'(x)$ and their sums are computationally inexpensive ($G'(x)$ is sparse, etc.). However, it is known that for a matrix $A \in \mathbb{R}^{n\times n}$ with $\rho(A) < 1$, additional conditions are needed in order that $A^k \to 0$ in floating point arithmetic (see [14, ch. 17] for a discussion of this topic).

Also, the local convergence of these iterations remains to be studied.

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