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SINGLE-VALUED AND MULTI-VALUED MEIR–KEELER TYPE OPERATORS

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. The purpose of this paper is to present several fixed point results for some single-valued and multi-valued Meir–Keeler type operators.

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1. INTRODUCTION

Let (X,d) be a complete metric space and $P(X)$ the space of all nonempty subsets of X. Denote by $P_{cp}(X)$ the space of all nonempty compact subsets of X. If H is the Hausdorff–Pompeiu metric on $P_{cp}(X)$, it is well known that $(P_{cp}(X),H)$ is a complete metric space.

Let f_i, $i\in\{1,\dots,m\}$ be a finite family of continuous single-valued operators of X into itself. We define the operator $T:(P_{cp}(X),H)\to(P_{cp}(X),H)$ by the following relation: $T(Y)=\bigcup\limits_{i=1}^{m} f_i(Y)$. If f_i are α-contractions for each $i\in\{1,\dots,m\}$ then the operator T is an α-contraction and hence has a unique fixed point.

On the other hand, if F_i, $i\in\{1,\dots,m\}$ is a finite family of upper semi-continuous multivalued operators, then the (single-valued) operator

$$T:(P_{cp}(X),H)\to(P_{cp}(X),H)\quad \text{given by } T(Y)=\bigcup_{i=1}^{m}F_i(Y)$$

is well defined. Moreover, it is well known that if F_i are α-contractions for

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each $i \in \{1, \ldots, m\}$ then T is an α-contraction too (see [6], [7]).

The purpose of this note is to prove that for each finite family of single-valued or multi-valued operators satisfying some Meir–Keeler type conditions the (single-valued) operator T has a fixed point.

2. PRELIMINARIES

Let (X, d) be a complete metric space and $P_{cp}(X)$ be the complete metric space of all nonempty, compact subsets of X. A metric space (X, d) is said to be ε-chainable (where $\varepsilon > 0$ is fixed) if and only if given $a, b \in X$ there is an ε-chain from a to b, that is a finite set of points $x_0, x_1, \ldots, x_n$ in X such that $x_0 = a$, $x_n = b$ and $d(x_{i-1}, x_i) < \varepsilon$, for all $i \in \{1, 2, \ldots, n\}$.

If $f : X \to X$ is a single-valued operator then $x^* \in X$ is a fixed point for f iff $x^* = f(x^*)$. We will denote by $Fixf$ the fixed points set of f.

If $F : X \to P(X)$ is a multi-valued operator then a fixed point for F is an element $x^* \in X$ such that $x^* \in F(x^*)$. The set of all fixed points for F will be denoted by Fix F.

Let us consider the following functionals :

$D : X \times P_{cp}(X) \to \mathbb{R}_+$, $D(x, A) = \inf\{d(x, a) | a \in A\}$, for $x \in X$

$\rho : P_{cp}(X) \times P_{cp}(X) \to \mathbb{R}_+$, $\rho(A, B) = \sup\{D(a, B) | a \in A\}$

$H : P_{cp}(X) \times P_{cp}(X) \to \mathbb{R}_+$, $H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.

Some contractivity-type conditions are needed in the main section.

Definition 2.1. *If $f : X \to X$ is an single-valued operator, let us consider the following conditions:*

i) α-contraction condition:

(1) there is $\alpha \in [0, 1[$ such that for $x, y \in X \;\Rightarrow\; d(f(x), f(y)) \le \alpha d(x, y)$

ii) strict contraction condition:

(2) $x, y \in X,\ x \ne y \;\Rightarrow\; d(f(x), f(y)) < d(x, y)$

iii) Meir–Keeler type condition:

(3) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$

iv) ε-locally Meir–Keeler type condition (where $\varepsilon > 0$)

(4) for each $0 < \eta < \varepsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta$ it follows $d(f(x), f(y)) < \eta$.

Let us observe that, condition (iii) implies (ii), (iii) implies (iv) and each of these conditions implies the continuity of f.

Definition 2.2. *If* $F : X \to P_{cp}(X)$ *is a multi-valued operator then* F *is said to be:*

i) α*-contraction if:*

(5) there is $\alpha \in [0, 1[$ *such that for* $x, y \in X \;\Rightarrow\; H(F(x), F(y)) \le \alpha d(x, y)$

ii) strict contraction if:

(6) $x, y \in X,\; x \ne y \;\Rightarrow\; H(F(x), F(y)) < d(x, y)$

iii) Meir–Keeler type operator if:

(7) for each $\eta > 0$ *there exists* $\delta > 0$ *such that* $x, y \in X,\ \eta \le d(x, y) < \eta + \delta \;\Rightarrow\; H(F(x), F(y)) < \eta$

iv) ε*-locally Meir–Keeler type operator (where* $\varepsilon > 0$*) if:*

(8) for each $0 < \eta < \varepsilon$ *there is* $\delta > 0$ *such that* $x, y \in X,\ \eta \le d(x, y) < \eta + \delta \;\Rightarrow\; H(F(x), F(y)) < \eta$*.*

It is easily to see that condition (iii) implies (ii), (iii) implies (iv) and each of these conditions implies the upper semi-continuity of F.

On the other hand, if $F : X \to P_{cp}(X)$ is an upper semi-continuous operator then $F(Y) \in P_{cp}(X)$ (see for example [1]).

Finally let us consider two fixed point principles given by Meir–Keeler [5] and Xu [8], that we need in the main section.

Theorem 2.1. *[5] Let* (X, d) *be a complete metric space and* f *an operator from* X *into itself. If* f *satisfies the Meir–Keeler type condition* (4) *then* f *has a unique fixed point, i.e.* $F_f = \{x^*\}$*. Moreover for any* $x \in X$*,* $\lim_{n\to\infty} f^n(x) = x^*$*.*

Theorem 2.2. *[8] Let* (X, d) *be a complete* ε*-chainable metric space and* $f : X \to X$ *be an operator satisfying the* ε*-locally Meir–Keeler type condition* (5)*. Then* f *has a fixed point.*

3. MAIN RESULTS

Let us consider first the single-valued operators $f_i : X \to X$, $i \in \{1, \dots, m\}$ and $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ the (single-valued) operator defined by the relation:

$$T(Y) = \bigcup_{i=1}^{m} f_i(Y). \tag{3.1}$$

Our first main result is:

THEOREM 3.1. *Let* (X,d) *be a complete metric space and* $f_i : X \to X$, *for* $i \in \{1,2,\ldots,m\}$ *are operators satisfying the Meir–Keeler type condition* (3). *Then the operator* $T : (P_{cp}(X),H) \to (P_{cp}(X),H)$ *defined by the relation* (3.1.) *is a Meir–Keller type operator and hence* $FixT = \{A^*\}$.

Proof. We shall prove that for each $\eta > 0$ there is $\delta > 0$ such that the following implication holds

$$\eta \leq H(A,B) < \eta + \delta \ \Rightarrow \ H(T(A),T(B)) < \eta.$$

Let us consider $A, B \in P_{cp}(X)$ such that $\eta \leq H(A,B) < \eta + \delta$.

If $u \in T(A)$ then there exists $j \in \{1,\ldots,m\}$ and $x \in A$ such that $u = f_j(x)$.

For $x \in A$ we can choose $y \in B$ such that $d(x,y) \leq H(A,B) < \eta + \delta$. We have the following alternative:

If $d(x,y) \geq \eta$ then $\eta \leq d(x,y) < \eta + \delta$ implies $d(f_j(x), f_j(y)) < \eta$. Hence $D(u,T(B)) \leq d(u, f_j(y)) < \eta$.

On the other hand, if $d(x,y) < \eta$ then from (3) we have $d(f_j(x), f_j(y)) < d(x,y) < \eta$ and again the conclusion $D(u,T(B)) < \eta$.

Because $T(A)$ is compact we have that $\rho(T(A),T(B)) < \eta$.

Interchanging the roles of $T(A)$ and $T(B)$ we obtain $\rho(T(B),T(A)) < \eta$ and hence $H(T(A),T(B)) < \eta$, showing the fact that T is a Meir–Keeler-type operator. From Meir–Keeler fixed point result (Theorem 2.1 below) we obtain that there exists an unique $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$. □

A fixed point result for a finite family of ε-locally single-valued Meir–Keeler type operators is:

THEOREM 3.2. *Let* (X,d) *be a complete* ε*-chainable metric space and* $f_i : X \to X$, *for* $i \in \{1,\ldots,m\}$ *be operators satisfying the* ε*-locally-Meir–Keeler type condition* (4). *Then the operator* $T : (P_{cp}(X),H) \to (P_{cp}(X),H)$ *defined by the relation* (3.1.) *is an* ε*-locally-Meir–Keeler type operator, having a fixed point.*

Proof. There are only minor modifications of the above arguments.The proof runs exactly as before, but instead of using the Meir–Keeler fixed point principle, the conclusion follows from Theorem 2.2. □

For the multi-valued case our main results are:

Theorem 3.3. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, for $i \in \{1, 2, \ldots, m\}$ are multi-valued Meir–Keeler type operators. Then the operator $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ defined by the relation:*

$$T(Y) = \bigcup_{i=1}^{m} F_i(Y). \tag{3.2}$$

is a (single-valued) Meir–Keller type operator, having a unique fixed point.

Proof. Let us suppose that for each $\eta > 0$ there exists $\delta > 0$ such that $\eta \le d(x, y) < \eta + \delta$ implies

$$H(F_i(x), F_i(y)) < \eta \text{ for } i \in \{1, \ldots, m\}. \tag{3.3}$$

It follows that F_i is contractive and hence F_i is upper semi-continuous, for $i \in \{1, \ldots, m\}$. As consequence $T : P_{cp}(X) \to P_{cp}(X)$.

Let us consider $\eta > 0$ and $Y_1, Y_2 \in P_{cp}(X)$ such that $\eta \le H(Y_1, Y_2) < \eta + \delta$. We will prove that $H(T(Y_1), T(Y_2)) < \eta$.

For this purpose, let $u \in T(Y_1)$ be arbitrary. Then there exist $k \in \{1, \ldots, m\}$ and $y_1 \in Y_1$ such that $u \in F_k(Y_1)$. For this $y_1 \in Y_1$ there is $y_2 \in Y_2$ such that $d(y_1, y_2) \le H(Y_1, Y_2) < \eta + \delta$.

If $d(y_1, y_2) \ge \eta$, then from (7) we get that $H(F_k(y_1), F_k(y_2)) < \eta$. It follows that there is $v \in F_k(y_2)$ such that $d(u, v) < \eta$ and hence $D(u, T(Y_2)) \le d(u, v) < \eta$.

On the other hand if $0 < d(y_1, y_2) < \eta$, then from the strict contraction condition we have that

$$H(F_k(y_1), F_k(y_2)) < d(y_1, y_2) < \eta$$

and as before $D(u, T(Y_2)) < \eta$.

Because $T(Y_1)$ is a compact set, we have that $\rho(T(Y_1), T(Y_2)) < \eta$. Interchanging the roles of $T(Y_1)$ and $T(Y_2)$ we obtain $\rho(T(Y_2), T(Y_1)) < \eta$ and the conclusion $H(T(Y_1), T(Y_2)) < \eta$ follows.

So $T : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator and by Theorem 2.1 has a unique fixed point, i.e. $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$. □

A local version of the previous result is:

Theorem 3.4. *Let (X, d) be a complete ε-chainable metric space (where $\varepsilon > 0$) and $F_i : X \to P_{cp}(X)$, for $i \in \{1, \ldots, m\}$ be a finite family of multi-valued ε-locally-Meir–Keeler type operators. Then the operator $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ defined by the relation* (3.2.) *is an ε-locally-Meir–Keeler type operator, having a fixed point.*

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