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CONVEXITY AND QUADRATIC MONOTONE APPROXIMATION IN DELAY DIFFERENTIAL EQUATIONS

RADU PRECUP

Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In this paper the method of quasiliniarization, an application of Newton's method, recently generalized in [1], is used for the quadratic, monotonic, bilateral approximation of the solution of the delay problem (5). The result is applied to an integral equation from biomathematics.

MSC 2000. 45L05, 65L05.

1. INTRODUCTION

In the papers [3]–[6] we investigated the following delay integral equation

$$x\left(t\right)=\int_{t-\tau}^{t}f\left(s,x\left(s\right)\right)ds \tag{1}$$

arising naturally from the study of the spread of virus diseases or, more generally, of the growth of single species populations. For example, we dealt with the initial values problem for (1) and looked for a continuous solution $x\left(t\right)$ of (1), for $-\tau\leq t\leq T$, when it is known that

$$x\left(t\right)=\varphi\left(t\right)\quad\text{for }\ -\tau\leq t\leq 0. \tag{2}$$

Obviously, we had to assume that

$$\varphi\left(0\right)=\int_{-\tau}^{0}f\left(s,\varphi\left(s\right)\right)ds. \tag{3}$$

Under assumption (3), problem (1)-(2) is equivalent with the following initial value problem

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$$(4)\qquad \begin{cases} x'(t) = f(t, x(t)) - f(t-\tau, \widetilde{x}(t-\tau)) & \text{for } 0 < t \leq T \\ x(0) = x_0 \end{cases}$$

where $x_0 = \varphi(0)$, $x \in C^1[0,T]$ and $\widetilde{x}(t) = \begin{cases} \varphi(t) & \text{for } -\tau \leq t \leq 0 \\ x(t) & \text{for } 0 < t \leq T \end{cases}$.

We obtained several existence and approximation results for the solutions of (4) by means of the monotone iterative method, assuming that $f(t,x)$ is monotone (increasing or decreasing) with respect to x. Such a result is the following one:

Suppose that $\varphi \in C[-\tau, 0]$ and let $u_0, v_0 \in C^1[0,T]$ such that $u_0(0) = v_0(0) = x_0$ and $u_0 < v_0$ on $(0,T]$. Define

$$\Omega = \{(t,x)\,; 0 < t \leq T,\ u_0(t) < x < v_0(t)\}$$

and

$$\widetilde{\Omega} = \overline{\Omega} \cup \{(t, \varphi(t))\,; -\tau \leq t < 0\}.$$

Proposition 1. *Assume*

(h1) *For $0 < t \leq T$, one has*

$$u_0'(t) \leq f(t, v_0(t)) - f(t-\tau, \widetilde{v}_0(t-\tau))$$

and

$$v_0'(t) \geq f(t, u_0(t)) - f(t-\tau, \widetilde{u}_0(t-\tau));$$

(h2) *$f \in C(\widetilde{\Omega})$, $f(t,.)$ is decreasing on $[u_0(t), v_0(t)]$ for each $t \in (0,T]$, and there is $L \geq 0$ such that*

$$|f(t,x) - f(t,y)| \leq L\,|x-y| \quad \textit{for } t \in [0,T] \textit{ and } x, y \in [u_0(t), v_0(t)].$$

Then (4) has a unique solution $x \in C^1[0,T]$ such that $u_0 \leq x \leq v_0$, and $x_n, y_n \to x$ uniformly on $[0,T]$, where $x_0 = u_0$, $y_0 = v_0$ and

$$x_n(t) = \int_{t-\tau}^{t} f(s, \widetilde{x}_{n-1}(s))\,ds, \quad y_n(t) = \int_{t-\tau}^{t} f(s, \widetilde{y}_{n-1}(s))ds.$$

Moreover, the sequences (x_{2n}) and (y_{2n+1}) are increasing, while (x_{2n+1}) and (y_{2n}) are decreasing.

Unfortunately, the convergence of the sequences (x_n) and (y_n) is only linear, more exactly

$$|x_n - x|_\infty \leq c\,|x_{n-1} - x|_\infty\,, \quad |y_n - x|_\infty \leq c\,|y_{n-1} - x|_\infty\,.$$

In this paper we prove that, if f is also convex, then there exist two monotone sequences (u_n) and (v_n) whose members are solutions of some linear equations, that converge quadratically to the unique solution of (4) from both directions. We say that the convergence of (u_n) and (v_n) is *quadratic* provided that

$$|u_n - x|_\infty\,,\ |v_n - x|_\infty \le c_1 \left|u_{n-1} - x\right|^2 + c_2 \left|v_{n-1} - x\right|^2.$$

We succeed this by adapting to (4) the recent quasilinearization method used in [1] for equations without delay. A second ingredient is the *step method* which is well known in the theory of delay equations.

2. RESULTS

We shall discuss a more general problem of type (4), namely

$$\begin{cases} x'(t) = f(t, x(t)) + g(t-\tau, \widetilde{x}(t-\tau)) & \text{for } 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{5}$$

THEOREM 1. *Assume*

(H1) *For $0 < t \le T$, one has*

$$u_0'(t) \le f(t, u_0(t)) + g(t-\tau, \widetilde{u}_0(t-\tau))$$

and

$$v_0'(t) \ge f(t, v_0(t)) + g(t-\tau, \widetilde{v}_0(t-\tau));$$

(H2) *$f \in C(\overline{\Omega})$, $g \in C(\widetilde{\Omega})$, the derivatives f_x, f_{xx}, g_x and g_{xx} exist and are continuous on $\overline{\Omega}$, and satisfy*

$$f_{xx} \ge 0, \quad g_x \ge 0 \quad \textit{and } g_{xx} \le 0 \quad \textit{on } \overline{\Omega}. \tag{6}$$

Then there exist the sequences (u_n) increasing and (v_n) decreasing which converge uniformly on $[0,T]$ to the unique solution $x \in C^1[0,T]$ of (5) satisfying $u_0 \le x \le v_0$, and the convergence is quadratic.

Proof. We use the convexity of f and concavity of g by means of the following two inequalities:

$$\begin{aligned} f(t,x) &\ge f(t,y) + f_x(t,y)(x-y), \\ g(t,x) &\ge g(t,y) + g_x(t,x)(x-y), \end{aligned} \tag{7}$$

which are true for all $(t,x), (t,y) \in \overline{\Omega}$.

Suppose we have already constructed the functions

$$u_0 \le u_1 \le \ldots \le u_n \le v_n \le \ldots \le v_1 \le v_0.$$

Then we take $u_{n+1} = \alpha$ and $v_{n+1} = \beta$, α and β being the unique solutions of the following linear initial value problems with delay

$$(8) \qquad \begin{cases} \alpha'(t) = F_n\left(t, \alpha(t), \widetilde{\alpha}(t-\tau)\right) & \text{for } 0 < t \leq T, \\ \alpha(0) = x_0, \end{cases}$$

respectively

$$(9) \qquad \begin{cases} \beta'(t) = G_n\left(t, \beta(t), \widetilde{\beta}(t-\tau)\right) & \text{for } 0 < t \leq T, \\ \beta(0) = x_0, \end{cases}$$

where

$$F_n = \begin{cases} f(t, u_n(t)) + f_x(t, u_n(t))(\alpha(t) - u_n(t)) + g(t-\tau, \widetilde{u}_n(t-\tau)) \\ \qquad + g_x(t-\tau, \widetilde{v}_n(t-\tau))(\widetilde{\alpha}(t-\tau) - \widetilde{u}_n(t-\tau)) \end{cases}$$

and

$$G_n = \begin{cases} f(t, v_n(t)) + f_x(t, u_n(t))(\beta(t) - v_n(t)) + g(t-\tau, \widetilde{v}_n(t-\tau)) \\ \qquad + g_x(t-\tau, \widetilde{v}_n(t-\tau))\left(\widetilde{\beta}(t-\tau) - \widetilde{v}_n(t-\tau)\right). \end{cases}$$

From (6) and (7) we easily see that the following inequalities hold:

$$\begin{aligned} u_n'(t) &\leq F_n(t, u_n(t), \widetilde{u}_n(t-\tau)), \\ v_n'(t) &\geq F_n(t, v_n(t), \widetilde{v}_n(t-\tau)). \end{aligned}$$

□

Next we need the following lemma:

LEMMA 1. *Let $u, v \in C^1[a,b]$ such that $u \leq v$, and let $H(t,x)$ be continuous for $t \in [a,b]$ and $u(t) \leq x \leq v(t)$. Suppose that*

$$u'(t) \leq H(t, u(t))$$

and

$$v'(t) \geq H(t, v(t))$$

on $[a,b]$. Then, for each $\alpha_0 \in [u(a), v(a)]$, there exists a solution $\alpha \in C^1[a,b]$ of the problem

$$\begin{cases} \alpha'(t) = H(t, \alpha(t)) \\ \alpha(a) = \alpha_0 \end{cases}$$

such that $u \leq \alpha \leq v$.

The proof of the Lemma can be given by using Corollary 3.1.2 in [2].

Now, we successively apply Lemma to the intervals $[0,\tau], [\tau, 2\tau], ..., [k\tau, T]$, where $k\tau < T \leq (k+1)\tau$. Thus we prove the existence of the solution $\alpha = u_{n+1}$ of (8) satisfying $u_n \leq \alpha \leq v_n$ on $[0,T]$. Similarly, we find a solution β of (9) such that $u_n \leq \beta \leq v_n$ on $[0,T]$.

Furthermore, since on $[0,T]$ one has

$$\alpha'(t) \le f(t,\alpha(t)) + g(t-\tau,\widetilde{\alpha}(t-\tau))$$

and

$$\beta'(t) \ge f(t,\beta(t)) + g(t-\tau,\widetilde{\beta}(t-\tau)),$$

by a comparison result (see Theorem 2.3 in [1]), and making use again of the step method, we can derive the inequality $\alpha \le \beta$, that is $u_{n+1} \le v_{n+1}$, on $[0,T]$.

Finally, by similar arguments, we can prove that the sequences (u_n) and (v_n) converge to the unique solution x, uniformly and quadratically.

Corollary 1. *Assume $\varphi \in C[-\tau,0]$, $u_0, v_0 \in C^1[0,T]$, $u_0(0) = v_0(0) = x_0$, $u_0 < v_0$ on $(0,T]$ and $f \in C(\widetilde{\Omega})$. In addition suppose that*

$$u_0'(t) \le f(t,u_0(t)) - f(t-\tau,\widetilde{u}_0(t-\tau))$$

and

$$v_0'(t) \ge f(t,v_0(t)) - f(t-\tau,\widetilde{v}_0(t-\tau)).$$

If the derivatives f_x and f_{xx} exist and are continuous on $\overline{\Omega}$, and

$$f_x \le 0, \quad f_{xx} \ge 0 \quad \text{on } \overline{\Omega},$$

then there exist the sequences (u_n)increasing and (v_n) decreasing which converge uniformly on $[0,T]$ to the unique solution x of (4) satisfying $u_0 \le x \le v_0$, and the convergence is quadratic.

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