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ON THE APPROXIMATION OF FUNCTIONS BY MEANS OF THE OPERATORS OF BINOMIAL TYPE OF TIBERIU POPOVICIU*

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In 1931, Tiberiu Popoviciu has initiated a procedure for the construction of sequences of linear positive operators of approximation. By using the theory of polynomials of binomial type (p_m) he has associated to a function $f \in C[0,1]$ a linear operator defined by the formula

$$(T_m f)(x) = \frac{1}{p_m(1)} \sum_{k=0}^{m} \binom{m}{k} p_k(x) p_{m-k}(1-x) f\left(\frac{k}{m}\right).$$

Examples of such operators were considered in several subsequent papers.

In this paper we present a convergence theorem corresponding to the sequence $(T_m f)$ and we also present a more general sequence of operators of approximation $S_{m,r,s}$, where r and s are nonnegative integers such that $2sr \leq m$.

We give an integral expression for the remainders, as well as a representation by using divided differences of second order.

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1. DEFINITION OF THE OPERATORS OF BINOMIAL TYPE T_m OF TIBERIU POPOVICIU

It is known that a sequence of polynomials (q_m), where m is a nonnegative integer, is said to be of **binomial type** if $deg\, q_m = m$, $p_0 \equiv 1$ and it obeys the following identities

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$$q_m(u+v) = \sum_{k=0}^{m} q_k(u) q_{m-k}(v), \tag{1.1}$$

for any nonnegative integer m.

The corresponding **sister sequence** (p_m), where $q_m = p_m/m!$, obeys the identities

$$p_m(u+v) = \sum_{k=0}^{m} \binom{m}{k} p_k(u) p_{m-k}(v). \tag{1.2}$$

By starting from the class of polynomials of binomial type (q_m), the great Romanian mathematician Tiberiu Popoviciu had already in 1931 [9] the wonderful idea to indicate a general method for construction linear positive operators, useful in the constructive theory of functions.

It is known that the sequence (q_m) is of binomial type if and only if it is defined by a generating relation

$$[\phi(t)]^x = \mathrm{e}^{x\varphi(t)} = \sum_{m=0}^{\infty} q_m(x) t^m,$$

where

$$\begin{aligned} \phi(t) &= 1 + a_1 t + a_2 t^2 + \ldots, \\ \varphi(t) &= c_1 t + c_2 t^2 + \ldots \quad (c_1 \neq 0) \end{aligned} \tag{1.3}$$

Selection $u = x$, $v = 1 - x$ suggested to Tiberiu Popoviciu to introduce an operator of binomial type, which associates to a function $f \in C[0,1]$ the polynomial

$$(T_m f)(x) = \frac{1}{a_m} \sum_{k=0}^{m} q_k(x) q_{m-k}(1-x) f\left(\frac{k}{m}\right),$$

which can be used for the approximation of the function f.

It is easy to see that, in fact, we have $a_m = q_m(1)$, where we suppose that $q_m(1) \neq 0$.

Taking this into account and the identity for the sister polynomials (p_m):

$$\sum_{k=0}^{m} \binom{m}{k} p_k(x) p_{m-k}(1-x) = p_m(1),$$

we can write

$$(T_m f)(x) = \frac{1}{p_m(1)} \sum_{k=0}^{m} \binom{m}{k} p_k(x) p_{m-k}(1-x) f\left(\frac{k}{m}\right). \tag{1.4}$$

We call it **the operator T_m of binomial type of Tiberiu Popoviciu**.

If we consider a result of T. Popoviciu [9], rediscovered later by P. Sablonnière [11], we have $p_\nu(x) \geq 0$ on $[0,1]$ ($\nu \in \mathbb{N}$), if and only if the coefficients c_k from (1.3) are non-negative. In this case the operator T_m is of positive type.

ILLUSTRATIVE EXAMPLES

A) If $p_m(x) = x^m$, we obtain the classical operator B_m of Bernstein

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\tfrac{k}{m}\right), \tag{1.5}$$
$$p_{m,k}(x) = \tbinom{m}{k} x^k (1-x)^{m-k}.$$

B) If we use the binomial polynomials represented by the factorial powers $p_m(x) = x^{[m,-\alpha]}$ ($\alpha \in \mathbb{R}_+$), then we get the operator S_m^α, defined by

$$(S_m^\alpha f)(x) = \sum_{k=0}^{m} w_{m,k}^\alpha(x) f\left(\tfrac{k}{m}\right), \tag{1.6}$$

where

$$w_{m,k}^\alpha(x) = \tbinom{m}{k} x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}/1^{[m,-\alpha]}. \tag{1.7}$$

The operator S_m^α was introduced in 1968 in our paper [13]. It was later investigated in several other papers [1], [2], [4], [7].

2. CONVERGENCE OF THE SEQUENCE $(T_m f)$

Assuming that all the coefficients c_k from (1.3) are non-negative, for the convergence of $(T_m f)$, where $f \in C[0,1]$, we can use the convergence criterion of Bohman-Popoviciu-Korovkin.

According to the identities satisfied by the binomial polynomials, we can see at once that we have: $T_m e_0 = e_0$.

In the case of the test function $f = e_1$, we get

$$(T_m e_1)(x) = p_m^{-1}(1) \sum_{k=0}^{m} \tfrac{k}{m} \tbinom{m}{k} p_k(x) p_{m-k}(1-x).$$

If we take into account that $\frac{k}{m}\binom{m}{k} = \binom{m-1}{k-1}$, we can write

$$(T_m e_1)(x) = p_m^{-1}(1) \sum_{k=1}^{m} \binom{m-1}{k-1} p_k(x) p_{m-k}(1-x).$$

It is easy to see that employing the change of index of summation $k-1=\nu$ and then denoting again the summation variable by k, we get

$$(T_m e_1)(x) = p_m^{-1}(1) \sum_{k=0}^{m-1} \binom{m-1}{k} p_{k+1}(x) p_{m-1-k}(1-x). \tag{2.1}$$

Because the binomial sequence (p_m) is generated by the following expansion

$$\mathrm{e}^{x\varphi(t)} = \sum_{m=0}^{\infty} p_m(x) \tfrac{t^m}{m!},$$

by differentiation we obtain

$$x\varphi'(t)\mathrm{e}^{x\varphi(t)} = p_1(x) + p_2(x)\frac{t}{1!} + \cdots + p_{m+1}(x)\frac{t^m}{m!} + \cdots.$$

It follows that we are able to write

$$\varphi'(t)\mathrm{e}^{x\varphi(t)} = \sum_{m=0}^{\infty} \tfrac{p_{m+1}(x)}{x} \cdot \tfrac{t^m}{m!}. \tag{2.2'}$$

We can now use the connection between sequences of polynomials of binomial type and sequences of Sheffer polynomials.

One says that a sequence of polynomials $(s_m)_{m\geq 0}$ is a **Sheffer sequence** for a theta operator θ if we have:

a) $s_0(x) = c \neq 0$; b) $\theta s_m(x) = m s_{m-1}(x)$.

Now we notice that a sequence (s_m) is a Sheffer sequence relative to a sequence of binomial type (p_m) if it satisfies the functional equation

$$s_m(u+v) = \sum_{k=0}^{m} \binom{m}{k} s_k(u) p_{m-k}(v). \tag{2.2}$$

A sequence of polynomials of Sheffer type is generated by an expansion similar with that connected with the binomial sequence [11], namely

$$\psi(t)e^{x\varphi(t)} = \sum_{m=0}^{\infty} s_m(t)\tfrac{t^m}{m!}, \quad \text{where } \psi(t) = \sum_{j=0}^{\infty} b_j t^j \quad (b_0 \neq 0). \tag{2.3}$$

Because at (2.2′) we have an expansion of the form (2.3), corresponding to the functions φ and $\psi = \varphi'$, in our case we have the Sheffer sequence $s_m(x) = p_{m+1}(x)/x$.

Imposing to this sequence to satisfy the equation (2.2), we get

$$\frac{p_{m+1}(u+v)}{u+v} = \sum_{k=0}^{m} \binom{m}{k} \frac{p_{k+1}(u)}{u} p_{m-k}(v).$$

If we decrease m by one and we choose $u = x$ and $v = 1 - x$, we find

$$x p_m(1) = \sum_{k=0}^{m-1} \binom{m-1}{k} p_{k+1}(x) p_{m-1-k}(1-x).$$

Replacing it in (2.1) we find that $T_m e_1 = e_1$. This result was found earlier by C. Manole [6] and then by P. Sablonnière [11].

Going on to the test function e_2, we mention two results.

1) The first one was found by C. Manole [6]:

$$(T_m e_2)(x) = x^2 + \frac{x(1-x)}{m} + x(1-x) a_m^{(2)}, \tag{2.4}$$

where

$$a_m^{(2)} = \tfrac{m-1}{m} [1 - p_m^{-1}(\theta') p_{m-2}(1)], \tag{2.4'}$$

θ' being the Pincherle derivative of the theta operator θ for which (p_m) is a basic sequence.

2) The second result belongs to P. Sablonnière [11]:

$$(T_m e_2)(x) = x^2 + x(1-x) b_m, \quad b_m = \frac{1}{m} + \frac{m-1}{m} \cdot \frac{r_{m-2}}{p_m}, \tag{2.5}$$

where $p_m = p_m(1)$ and $r_m = r_m(1)$, the sequence (r_m) being generated by the expansion

$$h''(t) \mathrm{e}^{x h(t)} = \sum_{m=0}^{\infty} r_m(x) \tfrac{t^m}{m!}.$$

Now we are able to state the following

THEOREM 2.1. *If $f \in C[0,1]$ and the operator T_m of Tiberiu Popoviciu is of positive type, then the sequence of polynomials $(T_m f)$ converges uniformly to the function f on the interval $[0,1]$ if we have:*

$$\lim_{m\to\infty} a_m^{(2)} = 0, \quad or \quad \lim_{m\to\infty} \frac{r_{m-2}}{p_m} = 0.$$

If we consider the approximation formula

$$f(x) = (T_m f)(x) + (R_m f)(x), \tag{2.6}$$

we can say that this formula has the degree of exactness equal to one.

On the other hand, it is easily to check that $x = 0$ and $x = 1$ are interpolation points of this polynomial, since $(T_m f)(0) = f(0), \quad (T_m f)(1) = f(1)$.

3. AN INTEGRAL REPRESENTATION OF THE REMAINDER

We can establish an integral form for the remainder of the approximation formula (2.6).

THEOREM 3.1. *If $f \in C^2[0,1]$ and x is a fixed point of the interval $[0,1]$ then the remainder of the Tiberiu Popoviciu approximation formula (2.6) can be represented under the following integral form*

$$(R_m f)(x) = \int_0^1 G_m(t;x) f''(t) dt, \tag{3.1}$$

where the Peano kernel $G_m(t;x)$ is given by the formula

$$G_m(t;x) = (T_m \varphi_x)(t), \quad \varphi_x(t) = \tfrac{1}{2}[x - t + |x - t|]. \tag{3.2}$$

Proof. The representation (3.1) can be obtained at once if we apply the well known theorem of Peano.

If we introduce the notation

$$w_{m,k}(x) = p_m^{-1}(1)\tbinom{m}{k} p_k(x) p_{m-k}(1-x),$$

we obtain

$$(R_m \varphi_x)(t) = (x-t)_+ - \sum_{k=0}^{m} w_{m,k}(x) \left(\tfrac{k}{m} - t\right)_+ . \tag{3.3}$$

□

Concerning the Peano kernel, we can state

THEOREM 3.2. *If we assume that $x \in \left[\frac{s-1}{m}, \frac{s}{m}\right]$ $(1 \le s \le m)$, the equation (3.2) permits to write the explicit formula*

$$G_m(t;x) = \begin{cases} -\sum\limits_{k=0}^{j-1} w_{m,k}(x)\left(t-\frac{k}{m}\right), & \textit{if} \quad t \in \left[\frac{j-1}{m}, \frac{j}{m}\right], \ 1 \le j \le s-1 \\ -\sum\limits_{k=0}^{s-1} w_{m,k}(x)\left(t-\frac{k}{m}\right), & \textit{if} \quad t \in \left[\frac{s-1}{m}, x\right] \\ -\sum\limits_{k=s}^{m} w_{m,k}(x)\left(\frac{k}{m}-t\right), & \textit{if} \quad t \in \left(x, \frac{s}{m}\right] \\ -\sum\limits_{k=j}^{m} w_{m,k}(x)\left(\frac{k}{m}-t\right), & \textit{if} \quad t \in \left(\frac{j-1}{m}, \frac{j}{m}\right], \ s < j \le m. \end{cases}$$

From these relations it is easy to see that on the square $D = [0,1] \times [0,1]$ we have $G_m(t;x) \le 0$.

Consequently, the equation $y = G_m(t) = G_m(t;x)$ represents a spline function of degree one, having the knots $\frac{k}{m}$ $(k = 0, 1, \ldots, m)$.

It may actually be shown that $G_m(t;x)$ represents the solution of a second-order differential system, under certain boundary conditions, so that it is the corresponding Green's function.

THEOREM 3.3. *If* $f \in C^2[0,1]$*, then the remainder of the T. Popoviciu approximation formula* (2.6) *can be represented by the following Cauchy type formula*

$$(R_m f)(x) = \tfrac{1}{2}(R_m e_2)(x) \cdot f''(\xi), \tag{3.4}$$

where $\xi \in (0,1)$.

Applying the first law of the mean to the integral (3.1) and replacing in the formula (2.6), we obtain

$$f(x) = (T_m f)(x) + f''(\xi) \int_0^1 G_m(t;x)dt.$$

If we substitute here $f(x) = e_2(x) = x^2$, we find that

$$\int_0^1 G_m(t;x)dt = \tfrac{1}{2}(R_m e_2)(x)$$

and we obtain formula (3.4).

4. REPRESENTATION OF THE REMAINDERS BY CONVEX COMBINATION OF SECOND-ORDER DIVIDED DIFFERENCES

Let $L_m : C[0,1] \to C[0,1]$ be a linear positive operator, defined by a formula of the following form

$$(L_m f)(x) = \sum_{k=0}^{m} q_{m,k}(x) f\left(\tfrac{k}{m}\right), \tag{4.1}$$

where on the interval $[0,1]$ we have $q_{m,k}(x) \geq 0$.

We assume that we have

$$(L_m f)(0) = f(0), \quad (L_m f)(1) = f(1)$$

and the formula

$$f(x) = (L_m f)(x) + (R_m f)(x) \tag{4.2}$$

has the degree of exactness $N = 1$.

As was shown in 1958 by I.J. Schoenberg, if L_m is not the identity operator then we have $L_m e_2 \neq e_2$, that is $R_m e_2 \neq 0$.

In many cases the remainder of the approximation formula (4.2) can be represented under the following form

$$(R_m f)(x) = (R_m e_2)(x) \cdot (D_m f)(x),$$

where $D_m f$ is a linear functional representing a convex combination of second-order divided differences of the function f on the point x and two consecutive nodes.

ILLUSTRATIONS

I. In 1964 we have proved that in the case of the Bernstein operator B_m we have

$$(R_m e_2)(x) = \frac{x(x-1)}{m}, \quad (D_m f)(x) = \sum_{k=0}^{m-1} p_{m-1,k}(x) \left[x, \tfrac{k}{m}, \tfrac{k+1}{m}; f\right].$$

II. In the case of the linear positive operator S_m^α, introduced in 1968 in our paper [13],

$$(S_m^\alpha f)(x) = \sum_{k=0}^{m} p_{m,k}^\alpha(x) f\left(\tfrac{k}{m}\right),$$

where

$$p^{\alpha}_{m,k}(x) = \binom{m}{k} x^{[k,-\alpha]}(1-x)^{[m-k,-\alpha]}/1^{[m,-\alpha]},$$

α being a parameter which might depend on m, the remainder of the corresponding approximation formula is

$$(R^{\alpha}_m f)(x) = (R^{\alpha}_m e_2)(x) \cdot (D^{\alpha}_m f)(x), \tag{4.3}$$

where

$$(R^{\alpha}_m e_2)(x) = \tfrac{1+\alpha m}{1+\alpha} \cdot \tfrac{x(1-x)}{m} \tag{4.4}$$

and

$$(D^{\alpha}_m f)(x) = \sum_{k=0}^{m-1} p^{\alpha}_{m-1,k}(x+\alpha, 1-x+\alpha) \left[x, \tfrac{k}{m}, \tfrac{k+1}{m}; f\right]. \tag{4.5}$$

Here we have used the notation

$$p^{\alpha}_{n,k}(u,v) = \binom{n}{k} u^{[k,-\alpha]} v^{[n-k,-\alpha]}/(u+v)^{[n,-\alpha]}.$$

III. In the case of the operators $S_{m,r,s}$, defined by the formula

$$(S_{m,r,s}f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \left[\sum_{j=0}^{s} p_{s,j}(x) f\left(\tfrac{k+jr}{m}\right)\right],$$

where r and s are nonnegative integer parameters satisfying the condition: $2sr \le m$, we have the approximation formula

$$f(x) = (S_{m,r,s}f)(x) + (T_{m,r,s}f)(x). \tag{4.6}$$

One observes that the polynomial $S_{m,r,s}f$ is interpolatory at both sides of the interval $[0,1]$, that is

$$(S_{m,r,s}f)(0) = f(0), \quad (S_{m,r,s}f)(1) = f(1).$$

By a straightforward calculation one can verify that if we consider the monomials $e_i(t) = t^i$ $(i = 0, 1, 2)$, where $t \in [0,1]$, then we obtain

$$S_{m,r,s}e_0 = e_0,\ S_{m,r,s}e_1 = e_1,\ (S_{m,r,s}e_2)(x) = x^2 + \left[1 + s\tfrac{r(r-1)}{m}\right] \cdot \tfrac{x(1-x)}{m}.$$

For the remainder of the approximation formula (4.6) one can find (see [15]) the following representation:

$$(4.7)\qquad (R_{m,r,s}f)(x) = \left[1 + s\tfrac{r(r-1)}{m}\right] \cdot \tfrac{x(x-1)}{m}(D_{m,r,s}f)(x),$$

where $(D_{m,r,s}f)(x)$ is given by the following convex combination of certain second-order divided differences of the function f on the point x and two consecutive nodes:

(4.8)

$$(D_{m,r,s}f)(x) = \tfrac{1}{m+sr(r-1)}\Bigg\{(m-sr)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x)\cdot \cdot\left(\sum_{j=0}^{s} p_{s,j}(x)\left[x, \tfrac{k+jr}{m}, \tfrac{k+jr+1}{m}; f\right]\right) + + sr^2\sum_{k=0}^{m-sr} p_{m-sr,k}(x)\left(\sum_{j=0}^{s-1} p_{s-1,j}(x)\left[x, \tfrac{k+jr}{m}, \tfrac{k+jr+r}{m}; f\right]\right)\Bigg\}.$$

One observes that all the coefficients of this linear functional are positive and their sum equals $(D_{m,r,s}e_2)(x) = 1$, for any $x \in [0,1]$. Hence it is made up by a convex combination of the second-order divided differences evidenced in the formula (4.8).

By using a theorem of T. Popoviciu [10] we can state

COROLLARY 4.1. *If $f \in C[0,1]$ and x is any fixed point of $[0,1]$, then there exist in this interval three distinct points u_m, v_m, w_m, which might depend on f, such that*

$$(R_{m,r,s}f)(x) = \left[1 + s\tfrac{r(r-1)}{m}\right] \cdot \tfrac{x(x-1)}{m}[u_m, v_m, w_m; f].$$

If we now apply the law of the mean for divided differences, we are able to state

COROLLARY 4.2. *If $f \in C^2[0,1]$ then there exists a point $\xi \in [0,1]$ such that the remainder can be expressed under the Cauchy form*

$$(R_{m,r,s}f)(x) = \left[1 + s\tfrac{r(r-1)}{m}\right] \cdot \tfrac{x(x-1)}{2m}f''(\xi).$$

In the special case $r = s = 0$ this formula was established in our paper [12].

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