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ON THE WHITE'S ALGORITHM FOR FRACTIONAL PROGRAMMING

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In this note we extend for fractional case a method due to White for solving a problem of maximizing over a finite set a function with some special "convexity" properties. Three algorithms applied to a transformation of the initial problem into a maximizing an auxilliary non-fractional function over a bi-product set are given.

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1. INTRODUCTION

This paper is motivated by the following class of fractional quadratic programming problems:

FQ. Find

$$\max_{x \in X} \frac{x'Cx + c'x + c_0}{x'Dx + e'x + e_0},$$

where X is a given (finite) subset of R^n, C is an $n \times n$ positive semidefinite matrix, D is an $n \times n$ negative semidefinite matrix, $c_0, e_0 \in R$,and c and e are given n-dimensional vectors, such that $x'Cx+c'x+c_0 > 0$ and $x'Dx+e'x+e_0 > 0$, for any $x \in X$.

The fractional quadratic programming problem **FQ** has various applications, e.g. fractional quadratic assignment problems of various kinds [8], nonlinear transportation problems, minimum risk stochastic programming problems [4], [6].

In this paper we extend some results obtained in [7] concerning the solving problem **FQ**.

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2. GENERAL PROBLEM FORMULATION

In this section we will consider a more general maximization problem as **FQ** and we will specify the functions involved in the statement of this programming problem and the feasible set. We will give also some properties of these classes of functions.

The feasible set X can be an arbitrary non-empty given set. In order to formulate the fractional maximization problem under study we consider the functions $g(\cdot,\cdot) : X \times X \to R$, $s(\cdot) : X \to R$, $d(\cdot) : X \to R$, $p(\cdot,\cdot) : X \times X \to R$, $f(\cdot) : X \to R$ with the following properties:

(1) $\quad f(x) = \dfrac{g(x,x) + s(x)}{p(x,x) + d(x)}$, for any $x \in X$,

(2) $\quad p(x,x) + d(x) > 0$ and $g(x,x) + s(x) \geq 0$, for any $x \in X$.

Next we consider the following fractional maximization problem.

FP. Find

$$\max_{x \in X} \frac{g(x,x) + s(x)}{p(x,x) + d(x)}.$$

In this paper we extend some results obtained in [7] concerning the solving of problem FP.

For $g(x,x) = x'Cx$, $s(x) = c'x + c_0$ and $p(x,x) = x'Dx$, $d(x) = e'x + e_0$,for any $x \in X$, problem FP becomes fractional quadratic programming problem FQ.

Definition 1. *[7] (i) The function g is called tr-convex (trace-convex) on $X \times X$ if g satisfies the inequality:*

(3) $\quad g(x,y) \leq \frac{1}{2}(g(x,x) + g(y,y))$, *for any* $(x,y) \in X \times X$.

(ii) The function g is called strictly tr-convex on $X \times X$ if g satisfies the inequality (3) and:

(4) $\quad g(x,y) < \frac{1}{2}(g(x,x) + g(y,y))$, *for any* $(x,y) \in X \times X, x \neq y$.

(iii) The function g is called tr-concave (strictly tr-concave) on $X \times X$ if $-g$ is tr-convex (strictly tr-convex) on X.

Definition 2. *[7]. (i) The function g is called tr-quasiconvex (trace-quasiconvex) on $X \times X$ if g satisfies the inequality:*

(5) $\quad g(x,y) \leq \max\{g(x,x), g(y,y)\}$, *for any* $(x,y) \in X \times X$.

(ii) The function g is called strictly tr-quasiconvex on $X \times X$ if g satisfies the inequality (3) and:

(6) $\quad g(x,y) < \max\{g(x,x), g(y,y)\}$, *for any* $(x,y) \in X \times X, x \neq y$.

(iii) The function g *is called tr-quasiconcave (strictly tr-quasiconcave) on* X *if* $-g$ *is tr-quasiconvex (strictly tr-quasiconvex) on* X.

It is simple to prove that if g is tr-convex (strictly tr-convex) then g is tr-quasiconvex (strictly tr-quasiconvex) too, but not conversely.

White [9], using functions g with properties (3) and (4), proposed two general algorithms for finding the optimal solution for maximizing a real function on a finite set.

We will associate to problem **FP**, for any $t \in R$, an extended non-fractional objective function $h(\cdot,\cdot,\cdot) : X \times X \times R \to R$ defined by

(7) $\quad h(x,y,t) = g(x,y) + \frac{s(x)+s(y)}{2} - t\left(p(x,y) + \frac{d(x)+d(y)}{2}\right)$, for any $(x,y,t) \in X \times X \times R$.

From (1) and (7), we have

(8) $\quad f(x) = \dfrac{h(x,x,t)}{p(x,x)+d(x)} + t$, for any $(x,t) \in X \times R$.

Lemma 1. *If* g *is a tr-convex (strictly tr-convex) and* p *is a tr-concave (strictly tr-concave) function on* $X \times X$, *then for any* $t \in R_+$ *the function* $h(\cdot,\cdot,t)$ *is tr-convex (strictly tr-convex) on* $X \times X$.

Proof. Indeed, from (7) and (3), h satisfies

$$\begin{aligned}
&h(x,x,t) + h(y,y,t) = \\
&= g(x,x) + s(x) - t(p(x,x) + d(x)) + g(y,y) + s(y) - t(p(y,y) + d(y)) \\
&\geq 2g(x,y) + s(x) + s(y) - 2tp(x,y) - t(d(x) + d(y)) \\
&= 2\left[g(x,y) + \tfrac{s(x)+s(y)}{2} - tp(x,y) - t\tfrac{d(x)+d(y)}{2}\right] \\
&= 2h(x,y,t),
\end{aligned}$$

for any $(x,y,t) \in X \times X \times R_+$.

Therefore, $h(\cdot,\cdot,t)$ is tr-convex on $X \times X$. For strictly tr-convexity the proof is similar. □

Lemma 2. *If* $t^* = f(x^*)$ *for some* $x^* \in X$, *then* $h(x^*,x^*,t^*) = 0$.

Proof. Indeed, from $t^* = f(x^*)$, using (1), we have

$$t^* = \frac{g(x^*,x^*) + s(x^*)}{p(x^*,x^*) + d(x^*)},$$

which is equivalent to

$$h(x^*, x^*, t^*) = g(x^*, x^*) + s(x^*) - t^*(p(x^*, x^*) + d(x^*)) = 0.$$

□

Let us define the following optimal sets:
(9) $X^* = \{x^* \in X | f(x^*) \geq f(x)$, for any $x \in X\}$,
(10) $X^*(t) = \{(x^*, y^*) \in X \times X | h(x^*, y^*, t) \geq h(x, y, t), \forall (x, y) \in X \times X\}$, for any $t \in R_+$,
(11) $F(x, t) = \{y^* \in X \mid h(x, y^*, t) \geq h(x, y, t)$, for all $y \in X\}$, for any $t \in R_+$.
The set X^* is the optimal set for problem **FP**, $X^*(t)$ is the optimal set for the maximization problem over the bi-product $X \times X$,

$$\max_{(x,y) \in X \times X} h(x, y, t),$$

and $F(x, t)$ is the optimal set of the problem

$$\max_{y \in X} h(x, y, t).$$

Then we have the following theorems.

Theorem 1. *Let g be tr-convex and p be tr-concave on $X \times X$. If $(x^*, y^*) \in X^*(t^*)$ and $t^* = f(x^*)$, then $x^* \in X^*$.*

Proof. Let $(x^*, y^*) \in X^*(t^*)$. Then, using (10), we have
(12) $h(x^*, y^*, t^*) \geq h(x, x, t^*)$, for any $x \in X$,
(13) $h(x^*, y^*, t^*) \geq h(y, y, t^*)$, for any $y \in X$,
Combining (12) and (13) with (8), and by using Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
(14) \qquad & f(x)(p(x, x) + d(x)) - t^*(p(x, x) + d(x)) + \\
& \quad + f(y)(p(y, y) + d(y)) - t^*(p(y, y) + d(y) \\
& = h(x, x, t^*) + h(y, y, t^*) \\
& \leq 2h(x^*, y^*, t^*) \\
& \leq h(x^*, x^*, t^*) + h(y^*, y^*, t^*) \\
& = g(x^*, x^*) - t^*(p(x^*, x^*) + d(x^*)) + s(x^*) + \\
& \quad + g(y^*, y^*) - t^*(p(y^*, y^*) + d(y^*)) + s(y^*) \\
& = 0 + g(y^*, y^*) - t^*(p(y^*, y^*) + d(y^*)) + s(y^*) \\
& = f(y^*)(p(y^*, y^*) + d(y^*)) - t^*(p(y^*, y^*) + d(y^*)),
\end{aligned}$$

for any $(x, y) \in X \times X$.

Setting $y = y^*$ in (14), we have

$$f(x)(p(x,x) + d(x)) - t^*(p(x,x) + d(x)) \le 0, \text{ for any } x \in X,$$

from where, since $p(x,x) + d(x) > 0$, it follows $f(x) \le t^*$, for any $x \in X$. This means that $x^* \in X$. □

Theorem 2. *Let g be tr-convex and p be tr-concave on $X \times X$ and $t^* = f(x^*)$. Then $x^* \in X^*$ if and only if $(x^*, y^*) \in X^*(t^*)$.*

Proof. (a) Let $x^* \in X$ and $(x,y) \in X \times X$. Then, we have
(15) $t^* = f(x^*) \ge f(x)$, for any $x \in X$,
(16) $t^* = f(x^*) \ge f(y)$, for any $y \in X$.
Since $p(x,x)+d(x) > 0$ and $p(y,y)+d(y) > 0$, from (15) and (16), it follows

$$h(x,x,t^*) = g(x,x) + s(x) - t^*(p(x,x) + d(x)) \le 0,$$
$$h(y,y,t^*) = g(y,y) + s(y) - t^*(p(y,y) + d(y)) \le 0,$$

from where, using Lemma 1, we obtain

$$2h(x,y,t^*) \le h(x,x,t^*) + h(y,y,t^*) \le 0 = 2h(x^*,x^*,t^*),$$

what means that $(x^*, y^*) \in X^*(t^*)$.
(b) Let $(x^*, y^*) \in X^*(t^*)$. Then, by Theorem 1, it follows that $x^* \in X^*$. □

Theorem 3. *Let g be tr-convex and p be tr-concave on $X \times X$ and $t^* = f(x^*)$. If $x^* \in X^*$ then $x^* \in F(x^*, t^*)$.*

Proof. Let $x^* \in X^*$. This means that

$$f(x^*) \ge f(x), \text{ for any } x \in X,$$

that is

$$t^* \ge \frac{g(x,x) + s(x)}{p(x,x) + d(x)}, \text{ for any } x \in X.$$

Hence, since $p(x,x) + d(x) > 0$, by Lemma 2, we obtain

$$\begin{aligned} h(x,x,t^*) &= g(x,x) + s(x) - t^*(p(x,x) + d(x)) \\ &\le 0 = h(x^*,x^*,t^*), \text{ for any } x \in X. \end{aligned}$$

Therefore, it results that x* ∈F(x*,t*). □

THEOREM 4. *Let g be tr-convex and p be tr-concave on $X \times X$, $x^* \in X$ and $t^* = f(x^*)$.*
(i) If $y^* \in F(x^*, t^*)$, then $f(y^*) \geq f(x^*)$.
(ii) If $x^* \in X^*$ and $y^* \in F(x^*, t^*)$, then $y^* \in X^*$.

Proof. (i) Let $y^* \in F(x^*, t^*)$. Then, using Lemma 1, we have

$$h(y^*, y^*, t^*) - h(x^*, y^*, t^*) \geq h(x^*, y^*, t^*) - h(x^*, x^*, t^*).$$

But $y^* \in F(x^*, t^*)$ implies $h(x^*, y^*, t^*) \geq h(x^*, x^*, t^*)$. Hence, we have

$$h(y^*, y^*, t^*) \geq h(x^*, y^*, t^*) \geq h(x^*, x^*, t^*) = 0.$$

This means that $g(y^*, y^*) + s(y^*) - t^*(p(y^*, y^*) + d(y^*)) \geq 0$, from where we obtain

$$f(x^*) = t^* \leq \frac{g(y^*, y^*) + s(y^*)}{p(y^*, y^*) + d(y^*)} = f(y^*). \tag{17}$$

(ii) On the other side, because $x^* \in X^*$ and $t^* = f(x^*)$, it follows that $t^* \geq f(y^*)$, which together with (17) implies that $y^* \in X^*$. □

3. EXTENDED WHITE'S ALGORITHMS

The following algorithms are inspired by White [9] and represent a combination between parametric algorithms [3], [5] for fractional programming and White's algorithms.

Algorithm **(FW)** for solving problem **FP** is as follows:

ALGORITHM 1 (FW).
Step 1. Select $x^1 \in X$ and set $k := 1$.
Step 2. Set $t^k = f(x^k)$.
Step 3. (i) If $F(x^k, t^k) \subseteq \{x^1, x^2, ..., x^k\}$ then stop. The optimal solution of FP is $x^* = x^k$.

(ii) If $F(x^k, t^k) \nsubseteq \{x^1, x^2, ..., x^k\}$, find

$$x^{k+1} \in F(x^k, t^k) - \{x^1, x^2, ..., x^k\}$$

and go to Step 4.

Step 4. Set $k := k + 1$ and go to Step 2.

Algorithm **FW** has the properties given in the following theorem.

Theorem 5. *Let g be tr-convex and p be tr-concave on $X \times X$.*
*(i) If algorithm **FW** generates at least two points, then*

$$f(x^{k+1}) \geq f(x^k), \ \ k \geq 1.$$

*(ii) If X is a finite set, algorithm **FW** terminates after a finite number of iterations.*
*(iii) If algorithm **FW** terminates after a finite number of iterations, then $x^k \in F(x^k, t^k)$.*

Proof. (i) Since, by Step 3(ii), $x^{k+1} \in F(x^k, t^k)$, and $t^k = f(x^k)$, the inequality $f(x^{k+1}) \geq f(x^k)$ follows by Theorem 4 (i).

(ii) This part follows from Step 3(ii) and the finiteness of X.

(iii) Let $x^* = x^k$ be a terminal point of the algorithm. From Step 3(i), we have

(18) $\qquad F(x^*, t^k) \subseteq \{x^1, x^2, ..., x^k\}$.

Then, from (18), for some $1 \leq i \leq k$, we have

(19) $\qquad h(x^*, x^i, t^k) \geq h(x^*, x^*, t^k), x^i \in F(x^*, t^k)$.

Since $h(\cdot, \cdot, t^k)$ is tr-convex from (1) it results

(20) $\qquad h(x^i, x^i, t^k) \geq h(x^*, x^i, t^k) \geq h(x^*, x^*, t^k)$.

From part (i) we have $t^k = f(x^k) \geq f(x^i)$, which implies

(21) $\qquad h(x^*, x^*, t^k) = 0 \geq h(x^i, x^i, t^k)$.

From (20) and (21) we have $h(x^*, x^i, t^k) = h(x^*, x^*, t^k)$. Therefore, it follows that $x^i \in F(x^k, t^k)$ and $x^k \in F(x^k, t^k)$. □

Algorithm **FW** at each iteration requires to keep all the previous obtained solutions as well as to make in Step 3 a lot of tests with these solutions.

Using Theorem 5 (i), we can improve algorithm **FW**, such at each iteration to memorize only a part of the previous obtained points.

Algorithm **FWM** for solving problem **FP** is as follows:

ALGORITHM 2 (FWM).

Step 1. Select $x^1 \in X$ and set $k := 1$ and $s := 1$.

In the algorithm, k represents the index of current iteration and s is the iteration index of the last modification of the function f, that is, the last iteration when $f(x^{s-1}) < f(x^s) = f(x^{s+1}) = ... = f(x^k)$.

Step 2. Set $t^k = f(x^k)$.

Step 3. (i) If $F(x^k, t^k) \subseteq \{x^s, x^{s+1}, ..., x^k\}$ then stop. The optimal solution of **FP** is $x^* = x^k$.

(ii) If $F(x^k, t^k) \nsubseteq \{x^s, x^{s+1}, ..., x^k\}$, find

$$x^{k+1} \in F(x^k, t^k) - \{x^s, x^{s+1}, ..., x^k\}$$

and go to Step 4.

Step 4. (i) If $f(x^{k+1}) > f(x^k)$, then set $k := k + 1$, $s := k + 1$ and go to Step 2.

(ii) If $f(x^{k+1}) = f(x^k)$, then set $k := k + 1$ and go to Step 3.

In order to obtain a better variant of algorithms **FW** or **FWM** next we consider that g is strictly tr-convex. The new variant of the algorithm will keep at each iteration only the current feasible solution obtained at previous iteration.

ALGORITHM 3 (FWS).

Step 1. Select $x^1 \in X$ and set $k := 1$.

Step 2. Set $t^k = f(x^k)$.

Step 3. (i) If $F(x^k, t^k) = \{x^k\}$ then stop. The optimal solution of **FP** is $x^* = x^k$.

(ii) If $F(x^k, t^k) \neq \{x^k\}$, find $x^{k+1} \in F(x^k, t^k) - \{x^k\}$ and go to Step 4.

Step 4. Set $k := k + 1$ and go to Step 2.

Algorithm **FWS** has the properties given in the following theorem.

THEOREM 6. *Let g be strictly tr-convex and p be tr-concave on $X \times X$.*

*(i) If algorithm **FWS** generates at least two points, then*

$$f(x^{k+1}) > f(x^k), k \geq 1.$$

*(ii) If X is a finite set, algorithm **FWS** terminates after k iterations.*

*(iii) If algorithm **FW** terminates after k iterations, then*

$$F(x^k, t^k) = \{x^k\}.$$

Proof. (i) Because x^{k+1} is generated in Step 3(ii), it follows that $x^{k+1} \neq x^k$, otherwise the algorithm should terminate in Step 3(i) with the optimal point x^k. Following the proof of Theorem 4(i), by using the strictly tr-convexity of g, we obtain in (17) a strict inequality and this strict inequality implies that $f(x^{k+1}) > f(x^k)$.

(ii) Since, by part (i), we have

$$f(x^1) < f(x^2) < ... < f(x^k) < f(x^{k+1}) < ...$$

it follows that algorithm **FWS** cannot cycle. Then, because X is finite, the requisite result follows.

(iii) This part of theorem follows immediately from Step 4(i) of the algorithm. □

In order to apply these algorithms to the fractional quadratic programming problem **FQ**, we state the following result that can be easily obtained by White [9] and Definitions 1 and 2.

THEOREM 7. *Let $g : X \times X \to R$ be such that $g(x, y) = x'Cy$ and $p(x, y) = x'Dy$ for any $(x, y) \in X \times X$. Then the following properties hold:*

(i) If C is positive semidefinite, then g is tr-convex on $X \times X$.

(ii) If C is positive definite, then gis strictly tr-convex on $X \times X$.

(iii) If D is negative semidefinite, then p is tr-concave on $X \times X$.

(iv) If D is negative definite, then p is strictly tr-concave on $X \times X$.

4. CONCLUSIONS

The algorithms **FW**, **FWM** and **FWS** can be interpreted as combinations between the White's algorithms [9] and parametric method for fractional programming [3],[5]. These algorithms are especially appropriate when X has a particular special form, such as assignment polytopes [8]. Otherwise, by a variable change of Charnes-Cooper type [2], the fractional quadratic programming problem could be reduced to a quadratic programming problem with an additional linear constraint, and this problem could be solved by an appropriate method.

We mention that an important class of fractional programming problems are obtained as equivalent deterministic problems of minimum risk programming problems [1], [4], [6].

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