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COMPACTNESS IN SPACES OF LIPSCHITZ FUNCTIONS

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. The aim of this paper is to prove a compactness criterium in spaces of Lipschitz and Fréchet differentiable mappings.

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1. INTRODUCTION

In the last years there have been an increasing interest in the study of Lipschitz functions and of spaces of Lipschitz functions, as a first step to extend to the nonlinear setting results from linear functional analysis. For instance, in the attempt to build a spectral theory for nonlinear operators, a special attention was paid to spectra of Lipschitz operators (see, e.g., [9], [2], [4]). Lipschitz duals, meaning spaces of Lipschitz functions on a metric linear space, were used to study best approximation problems in such spaces (see [10]). A good account on Banach spaces and Banach algebras of Lipschitz functions is given in the monograph [11]. The monograph [6] contains a comprehensive study of Lipschitz functions on Banach spaces and their applications to the geometry of Banach spaces (e.g. the Lipschitz classification of Banach spaces).

As asserts Appell [1], apparently there is no compactness criterium in spaces of Hölder functions, and some criteria given in the literature turned to be false (e.g. that in [7]). The aim of this Note is to prove such a criterium (a true one, I hope) for families of Lipschitz and Fréchet differentiable mappings. The paper by J. Batt [5] contains a detailed study of compactness for nonlinear mappings and their adjoints, including Schauder type theorems. A Schauder type theorem for differentiable mappings was proved also by Yamamuro [12].

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2. THE RESULT

Let X, Y be real or complex normed linear spaces, and Ω a subset of X. Denote by $\mathrm{Lip}(\Omega, Y)$ the space of all Lipschitz mappings from Ω to Y, i.e. those mappings $f : \Omega \to Y$ for which the number

$$L(f) := \sup\{\|f(x) - f(y)\|/\|x - y\| : x, y \in \Omega,\ x \neq y\} \tag{1}$$

is finite. The number $L(f)$ defined by (1) is called the Lipschitz norm of the mapping f, and it is the smallest Lipschitz constant for f. The function $L(\cdot)$ is a seminorm on $\mathrm{Lip}(\Omega, Y)$, so that $(\mathrm{Lip}(\Omega, Y), L)$ is a seminormed space which is complete if Y is a Banach space. (The operations of addition and multiplication by scalars are defined pointwisely)

If Ω is an open subset of X, denote by $C^1(\Omega, Y)$ the space of all continuously Fréchet differentiable mappings from Ω to Y, and for $K \subset \Omega$ put

$$C^1 \mathrm{Lip}(K, Y) := \{f \in \mathrm{Lip}(K, Y) : \exists F \in C^1(\Omega, Y) \quad \text{such that} \quad F|_K = f\}.$$

Let also $L(X, Y)$ denote the space of all continuous linear operators from X to Y equipped with the uniform norm.

The compactness result we shall prove is the following:

Theorem 1. *Let X, Y be normed spaces, Ω an open subset of X and K a compact convex subset of Ω.*

Suppose that Z is a subset of $C^1 \mathrm{Lip}(K, Y)$ such that

(i) *for every $x \in K$ the set $\{f'(x) : f \in Z\}$ is totally bounded in $L(X, Y)$;*
(ii) *for every $x \in K$ and every $\epsilon > 0$ there exists $\delta = \delta(x, \epsilon) > 0$ such that*

$$\forall x' \in B(x, \delta) \subset \Omega,\ \forall f \in Z \quad \|f'(x) - f'(x')\| \leq \epsilon.$$

Then the set Z is totally bounded in $\mathrm{Lip}(K, Y)$.

Conversely, if the set $Z \subset C^1 \mathrm{Lip}(\Omega, Y)$ is totally bounded in $\mathrm{Lip}(\Omega, Y)$ then Z satisfies the conditions (i) and (ii).

As consequence, one obtains the following corollary.

Corollary 1. *If Y is a Banach space and $Z \subset C^1 \mathrm{Lip}(K, Y)$ is closed and satisfies the conditions (i) and (ii) from Theorem 1 then the set Z is compact in $\mathrm{Lip}(K, Y)$.*

The proof of Theorem 1 will be based on the following lemma:

Lemma 1. *Let X, Y be normed spaces and Ω an open subset of X. If $g : \Omega \to Y$ satisfies*

$$\|g(x_0) - g(x)\| \le \lambda \|x_0 - x\|, \tag{2}$$

for every x in a neighborhood $U \subset \Omega$ of x_0 and g is Fréchet differentiable at x_0, then

$$\|g'(x_0)\| \le \lambda. \tag{3}$$

Conversely, if g is Fréchet differentiable on an open convex neighborhood $U \subset \Omega$ of x_0 and

$$\|g'(x)\| \le \lambda, \qquad \forall x \in U, \tag{4}$$

then

$$\|g(x) - g(y)\| \le \lambda \|x - y\|, \qquad \forall x, y \in U. \tag{5}$$

Proof of Lemma 1. Suppose that $g : \Omega \to Y$ satisfies (2). The differentiability of g at x_0 implies the existence of $g'(x_0) \in L(X, Y)$ such that

$$g(x_0 + h) - g(x_0) = g'(x_0)h + \|h\|\alpha(h), \tag{6}$$

where $\lim_{h\to 0} \alpha(h) = 0$. For $n \in \mathbb{N}$ choose $\delta_n > 0$ such that $\overline{B}(x_0, \delta_n) \subset \Omega$ and

$$\|\alpha(h)\| \le 1/n, \qquad \forall h \in \overline{B}(0, \delta_n).$$

Then, from (6),

$$\begin{aligned}\|g'(x_0)\| &\le \|g(x_0 + h) - g(x_0)\| + \|h\|\|\alpha(h)\| \\ &\le (\lambda + \tfrac{1}{n})\|h\|.\end{aligned}$$

The inequality

$$\|g'(x_0)h\| \le (\lambda + \tfrac{1}{n})\|h\|, \quad \forall h,\ \|h\| \le \delta_n,$$

implies $\|g'(x_0)\| \le \lambda + 1/n$, $\forall n \in \mathbb{N}$, so that $\|g'(x_0)\| \le \lambda$.

Conversely, suppose that g is Fréchet differentiable on an open convex neighborhood $U \subset \Omega$ of x_0, and satisfies (4).

By the mean value theorem

$$\|g(x) - g(y)\| \le \|x - y\| \sup\{\|g'(\xi)\| : \xi \in [x, y]\} \le \lambda \|x - y\|,$$

for all $x, y \in U$.

Lemma 1 is proved. □

Proof of Theorem 1.

Suppose that the set $Z \subset C^1 \operatorname{Lip}(K, Y)$ satisfies the conditions (i) and (ii), and let $\epsilon > 0$ be given.

By (ii), for every $x \in K$ there exists $\delta_x > 0$ such that

$$\forall f \in Z \quad \text{and} \quad \forall x' \in B(x, \delta_x) \cap K \quad \|f'(x) - f'(x')\| \leq \epsilon. \tag{7}$$

Since the set K is compact, there exists $x_1, \dots, x_p$ in K such that

$$K \subset \bigcup_{k=1}^{p} B(x_k, \delta_k), \quad \text{where} \quad \delta_k = \delta_{x_k}. \tag{8}$$

By (i), the set $Y_k = \{f'(x_k) : f \in Z\}$ is totally bounded in $L(X, Y)$, for $k = 1, 2, \dots, p$. It follows that the set

$$W = Y_1 \times \cdots \times Y_k$$

is totally bounded in $(L(X, Y))^p$ with respect to the norm

$$\|(A_1, \dots, A_p)\| = \max\{\|A_1\|, \dots, \|A_p\|\},$$

as well as the set

$$H = \{(f'(x_1), ..., f'(x_p)) : f \in Z\} \subset W.$$

Therefore we can find $f_1, \dots, f_n$ in Z such that

$$\forall f \in Z \; \exists j \in \{1, \dots, n\} \quad \text{such that} \quad \|f'(x_k) - f_j'(x_k)\| \leq \epsilon, \tag{9}$$

for $k = 1, \dots, p$.

We shall show that $\{f_1, \dots, f_n\}$ is a 3ϵ-net for the set Z with respect to the Lipschitz norm (1) on $\operatorname{Lip}(K, Y)$.

Let $f \in Z$. By (9) there is $j \in \{1, \dots, n\}$ such that

$$\|f'(x_k) - f_j'(x_k)\| \leq \epsilon, \quad \text{for} \quad k = 1, \dots, n. \tag{10}$$

By the mean value theorem, we have for $x, y \in K$

$$\|(f - f_j)(x) - (f - f_j)(y)\| \leq \|x - y\| \sup\{\|(f' - f_j')(\xi)\| : \xi \in [x, y]\}. \tag{11}$$

Since $\xi \in [x, y] \subset K$, by (8) there exists $k \in \{1, \dots, p\}$ such that $\xi \in B(x_k, \delta_k)$. But then

$$\|(f' - f_j')(\xi)\| \leq \|f'(\xi) - f'(x_k)\| + \|(f' - f_j')(x_k)\| + \|f_j'(x_k) - f_j'(\xi)\| \leq 3\epsilon. \tag{12}$$

(We have applied (7) to the first and the last term, and (10) to the middle one).

By (11) and (12)

$$\|(f - f_j)(x) - (f - f_j)(y)\| \leq 3\epsilon \|x - y\|$$

for all $x, y \in K$ or, equivalently,

$$L(f - f_j) \le 3\epsilon.$$

To prove the converse implication, suppose that $Z \subset \text{Lip}(\Omega, Y) \cap C^1(\Omega, Y)$ is totally bounded in $\text{Lip}(\Omega, Y)$, and let $\epsilon > 0$ be given. Choose an ϵ-net $\{f_1, \dots, f_n\} \subset Z$, i.e. $\forall f \in Z\ \exists j \in \{1, \dots, n\}$ such that $\forall x, y \in \Omega$:

$$\|(f - f_j)(x) - (f - f_j)(y)\| \le \epsilon \|x - y\|.$$

Taking into account Lemma 1, one obtains

$$\|f'(x) - f_j'(x)\| \le \epsilon,$$

for all $x \in \Omega$, showing that $\{f_1'(x), \dots, f_n'(x)\}$ is an ϵ-net for the set $\{f'(x) : f \in Z\}$. Therefore (i) holds.

To prove (ii), let $\epsilon > 0$ and $x \in \Omega$ be fixed. Choose again an ϵ-net $\{f_1, \dots, f_n\}$ for the set Z. Since the mappings f_i are of class C^1 there exists $\delta > 0$ such that

$$\forall x' \in B(x, \delta) \subset \Omega \quad \text{and} \quad \forall i \in \{1, \dots, n\} \quad \|f_i'(x) - f_i'(x')\| \le \epsilon. \tag{13}$$

For $f \in Z$ choose $j \in \{1, \dots, n\}$ such that

$$L(f - f_j) \le \epsilon. \tag{14}$$

By Lemma 1 this implies

$$\forall y \in \Omega \quad \|(f' - f_j')(y)\| \le \epsilon. \tag{15}$$

Taking into account (13) and (15), we obtain

$$\|f'(x) - f'(x')\| \le \|f'(x) - f_j'(x)\| + \|f_j'(x) - f_j'(x')\| + \|f_j'(x') - f'(x')\| \le 3\epsilon$$

for all $x' \in B(x, \delta)$, which shows that (ii) holds too.

Theorem 1 is completely proved. □

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