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UNIVARIATE SHEPARD-BIRKHOFF INTERPOLATION

GHEORGHE COMAN and RADU T. TRÎMBIŢAŞ

Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In these paper we study the univariate Shepard-Birkhoff operators. We give an error estimation for the aproximation with these operators and some examples and graphs.

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1. INTRODUCTION

Given a set of $n+1$ pairwise distinct points $X = \{x_0, x_1, \ldots, x_n\} \subset I = [a,b], a, b \in \mathbb{R}, a < b$, the classical Shepard interpolation operator [13] given by

$$\left(S^0_{n,\mu} f\right)(x) = \sum_{k=0}^{n} w_k(x) f(x_k) \tag{1}$$

$$w_k(x) = \frac{|x-x_k|^{-\mu}}{\sum\limits_{k=0}^{n} |x-x_k|^{-\mu}}, \tag{2}$$

has a low degree of exactness.

Remark 1. The form of weights functions given by (2) is the barycentric form; it is not suitable when $x = x_k$ or when x is close to x_k. In these cases the following form is more convenient:

$$w_k(x) = \prod_{\substack{j=0\\ j\neq k}} |x-x_k|^{\mu} \Big/ \sum_{k=0}^{n} \prod_{\substack{j=0\\ j\neq k}} |x-x_k|^{\mu}. \tag{3}$$

"Babeş–Bolyai" University, Faculty of Mathematics and Informatics, str. Kogălniceanu 1, 3400 Cluj–Napoca, Romania, e-mail: tradu@math.ubbcluj.ro, ghcoman@math.ubbcluj.ro.

In order to increase the degree of exactness one tries to replace the values $f(x_k)$ with the values of an interpolation operator: Taylor [2], [11], [3], [6], [4], Lagrange ([5], [7] for the multivariate case and [14] for the univariate case), Hermite [8], [7], Birkhoff [9], [7].

In this paper we shall consider the following univariate Shepard-type operator

$$\left(S_{n,\mu}^{B;q} f\right)(x) = \sum_{k=0}^{n} w_k(x)(B_q f)(x; x_k), \tag{4}$$

where $(B_q f)(x; x_k)$ is a suitable q-th degree polynomial interpolatory operator (in our case a Birkhoff-type operator, see):

$$(B_m f)(x) = \sum_{j=0}^{p} \sum_{i \in I_j} b_{j,p}(x) f^{(j)}(x_{k+j}), \tag{5}$$

$r_j \in \mathbb{N}$ and $I_j \subset \{0, \ldots, r_j\}$, $j = \overline{0, n}$.

Remark 2. We suppose $(B_q f)(x; x_k)$ exists.

If $(B_q f)(x; x_k)$ in (4) is the Taylor polynomial

$$(T_q f)(x; x_k) = \sum_{\nu=0}^{q} \frac{1}{\nu!} f^{(\nu)}(x_k)(x - x_k)^{\nu} \tag{6}$$

we obtain the Shepard-Taylor operator

$$\left(S_{n,\mu}^{B;q} f\right)(x) := \left(S_{n,\mu}^{q} f\right)(x) = \sum_{k=0}^{n} \sum_{\nu=0}^{q} \frac{1}{\nu!} f^{(\nu)}(x_k)(x - x_k)^{\nu} w_k(x). \tag{7}$$

This operator was investigated in [1], [2], [4], [3] for the univariate case and in [11], [6] for the multivariate case.

We set

$$\begin{aligned} B_\rho(x) &= [x - \rho, x + \rho], \\ r &= \inf\{\rho > 0 : \forall x \in A, \exists u \in X \ u \in B_\rho(x)\}, \quad \text{and} \\ M &= \sup_{y} card(B_r(Y) \cap X) \end{aligned}$$

(i.e. M is the maximum number of points from X contained in an interval $B_r(x)$).

2. ERROR ESTIMATIONS

First we give an error estimation for the Shepard-Taylor operator given by (7).

THEOREM 3. *If $f \in C^{q+1}(I)$, then*

$$\left\|S^q_{n,\mu} f - f\right\|_I \leq CM \left\|f^{(q+1)}\right\|_I \varepsilon^q_\mu(r), \tag{8}$$

where[1]

$$\varepsilon^q_\mu(r) = \begin{cases} |\log r|^{-1}, & \mu = 1 \\ r^{\mu-1}, & 1 < \mu < q+2 \\ r^{\mu-1}|\log r|, & \mu = q+2 \\ r^{q+1}, & \mu > q+2. \end{cases} \tag{9}$$

and C is a positive constant independent of x and X. If $f \in C(I)$, then

$$\left\|S^q_{n,\mu} f - f\right\|_I \leq CM\omega(f; \varepsilon^0_\mu(r)).$$

Proof. We define

$$s^q_\mu(x) = \sum_{k=0}^{n} |x - x_k|^{q+1-\mu} \Big/ \sum_{k=0}^{n} |x - x_k|^{-\mu}.$$

We have

$$|(R_q f)(x; x_k)| = |(T_q f)(x; x_k) - f(x)| = c_q |x - x_k| ||f^{(q+1)}||_I,$$

where $c_p = 1/(p+1)!$ for $p > 0$ or $c_p = 1$ for $p = 0$.

If $f \in C^{q+1}(I)$, then

$$\left|S^q_{n,\mu} f(x) - f(x)\right| \leq \sum_{k=0}^{n} \left|(R_q f)(x; x_k)\right| w_k(x) \leq C^q \left\|f^{(q+1)}\right\|_I s^q_\mu(x).$$

We shall show that

$$s^q_\mu(x) \leq MC\varepsilon^q_\mu(r)$$

where C is a positive constant independent of x and X.

Let be $N = [(b-a)/(2r)] + 1$, $Q_r(u)$ be the interval $(u - r, u + r]$, and $T_j = Q_r(x + 2rj) \cup Q_r(x + 2rj)$.

The set

[1] $\|f\|_A$ means $\sup\{|f(x)| : x \in A\}$

$$\bigcup_{j=-N}^{N} Q_r(x+2rj)$$

constitutes a disjoint covering of I with half-open intervals. Obviously, for $x_k \in I \cap T_j$ we have

$$(2j-1)r \le |x-x_k| \le (2j+1)r, \quad j=\overline{1,N}$$

and

$$1 \le card(X \cap T_j) \le M.$$

Also, $n = cardX = O(r^{-1})$.

Defining x_d as being the closest point to x and since

$$\sum_{k=0}^{n} |x-x_k|^{\mu} \ge |x-x_d|^{-\mu}$$

we get

$$\begin{aligned} s_{\mu}^{q}(x) &\le |x-x_d|^{\mu} \left(\sum_{x_k \in T_0} |x-x_k|^{q+1-\mu} + \sum_{j=1}^{N} \sum_{x_k \in T_j} |x-x_k|^{q+1-\mu} \right) \\ &\le r^{\mu} \left(Mr^{q+1-\mu} + M \sum_{j=0}^{N} [(2j+1)r]^{q+1-\mu} \right) \\ &\le Mr^{q+1} \left(1 + C \sum_{j=1}^{N} j^{q+1-\mu} \right). \end{aligned}$$

Case 1. ($\mu > 1$)
subcase 1a. If $1 < \mu < q+2$, then

$$r^{q+1}(1 + C \sum_{j=1}^{N} j^{q+1-\mu}) = O(r^{\mu-1}).$$

subcase 1b. If $\mu = q+2$, then

$$\sum_{j=1}^{N} j^{q+1-\mu} = \log N = |\log r|.$$

subcase 1c. If $\mu > q+2$, then $\sum_{j=1}^{N} j^{q+1-\mu}$ is bounded.

Case 2. ($\mu = 1$) The function s_{μ}^{q} becomes

$$s_1^q(x) = \sum_{k=0}^{n} |x-x_k|^{q} \Big/ \sum_{k=0}^{n} |x-x_k|^{-1}. \tag{10}$$

The numerator in (10) is less than or equal to

$$\sum_{x_k \in T_0} |x - x_k|^q + \sum_{j=1}^{N} \sum_{x_k \in T_j} |x - x_k|^q,$$

and the denominator is greater than or equal to

$$\sum_{x_k \in T_0} |x - x_k|^{-1} + \sum_{j=1}^{N} \sum_{x_k \in T_j} |x - x_k|^{-1} \geq \sum_{x_k \in T_0} |x - x_k|^{-1} + \tfrac{C}{r} |\log r|.$$

Applying the inequality

$$\frac{\sum a_i}{\sum b_i} \leq \sum \tfrac{a_i}{b_j}$$

we get

$$\begin{aligned} s_1^q(x) &\leq \sum_{x_k \in T_0} |x - x_k|^{q+1} + C_1 \tfrac{r}{|\log r|} \sum_{j=1}^{N} \sum_{x_k \in T_j} |x - x_k|^q \\ &\leq M r^{q+1} \left(1 + C_2 \frac{1}{|\log r|} \sum_{j=1}^{N} j^q \right) \\ &\leq M r^{q+1} \left(1 + \frac{C_2}{|\log r|} O(r^{-q-1}) \right) \\ &= O(|\log r|^{-1}). \end{aligned}$$

If $f \in C(I)$, setting $\varepsilon(r) = \varepsilon_\mu^0(r)$, since

$$\omega(f;\delta) = \omega\left(f; \frac{\delta}{\varepsilon}\right) \leq \left(1 + \frac{\delta}{\varepsilon}\right) \omega(f;\varepsilon) \leq \frac{C}{\varepsilon} \omega(f;\varepsilon),$$

we have successively

$$\begin{aligned} \left| S_{n,\mu}^q f(x) - f(x) \right| &\leq \sum_{|x - x_k| < \varepsilon(r)} \omega(f;\varepsilon(r)) w_k(x) + \sum_{\substack{k \\ |x - x_k| > \varepsilon(r)}} |f(x) - f(x_k)| w_k(x) \\ &\leq \omega(f;\varepsilon(r)) \left(1 + \frac{C}{\varepsilon(r)} s_\mu^0(x)\right). \end{aligned}$$

□

The next theorem gives an error estimation for the Shepard-Birkhoff operator given by (5)

THEOREM 4. *If* $f \in C^{q+1}(I)$, *then*

$$\left\| S_{n,\mu}^{B,q} f - f \right\|_I \leq DM \left\| f^{(q+1)} \right\|_I \varepsilon_\mu^q(r),$$

where D *is a constant independent of* x *and* X.

Proof. Since B_q is a projector, we have

$$\begin{aligned} ||f - B_q f|| &\le ||f - T_q f|| + ||T_q f - B_q f|| \\ &\le ||f - T_q f|| + ||B_q T_m f - B_q f|| \\ &\le (1 + ||B_q||)\, ||f - T_q f||, \end{aligned}$$

and

$$\begin{aligned} \left|S^{B,q}_{n,\mu} f(x) - f(x)\right| &\le \left|S^{B,q}_{n,\mu} f(x) - S^{q}_{n,\mu} f(x)\right| + \left|S^{q}_{n,\mu} f(x) - f(x)\right| \\ &\le \left|\sum_{k=1}^{n} w_k(x)\left((B_q f)(x) - (T_q f)(x)\right)\right| + \\ &\quad + \left|\sum_{k=1}^{n} w_k(x)\left(f(x) - (T_q f)(x)\right)\right| \\ &\le (1 + ||B_q||)\left|\sum_{k=1}^{n} w_k(x)\left(f(x) - (T_q f)(x)\right)\right|. \end{aligned}$$

Applying now the Theorem 3, the conclusion follows immediately. □

3. PARTICULAR CASES

The existence and uniqueness of Birkhoff interpolation polynomial is not assured for the general case. We shall considered some particular cases when the existence and the uniqueness of Birkhoff interpolation polynomial is guaranteed and the corresponding Shepard-type operators.

The Birkhoff interpolation has as particular cases the Hermite interpolation, the Lagrange interpolation and the Taylor interpolation.

The Shepard-Taylor interpolation was treated in sections **??** and **??**. Let's consider now the Shepard-Hermite and Shepard-Lagrange interpolation.

We put $x_{n+k} = x_{n-m+k-1}$, for $k = \overline{1,n}$.

In the case of Shepard-Hermite interpolation for the sake of simplicity we choose node multiplicities equal: $r_0 = r_1 = \ldots = r_n = r$. Thus we obtain the following operator

$$\left(S^{H,q}_{n,\mu} f\right)(x) = \sum_{k=0}^{n} w_k(x)(H_q f)(x; x_k),$$

where

$$(H_q f)(x; x_k) = \sum_{j=0}^{m} \sum_{p=0}^{r} h_{j,p}(x) f^{(p)}(x_{k+j}),$$

$h_{j,p}$ being the basic Hermite's polynomial.

The Shepard-Lagrange interpolation operator has the form

$$\left(S_{n,\mu}^{L,q} f\right)(x) = \sum_{k=0}^{n} w_k(x)(L_q f)(x; x_k),$$

where

$$(L_q f)(x; x_k) = \sum_{k=0}^{n} l_{j,k}(x) f(x_{k+j})$$

and $l_{k,j}$ are the basic Lagrange's polynomials, i.e.

$$l_{j,k}(x) = \frac{(x - x_k)\dots(x - x_{k+j-1})(x - x_{k+j-1})\dots(x - x_{k+q})}{(x_{k+j} - x_k)\dots(x_{k+j} - x_{k+j-1})(x_{k+j} - x_{k+j-1})\dots(x_{k+j} - x_{k+q})}.$$

Finally, we give an example based on Abel-Gontcharoff interpolation (see [10]):

$$\left(S_{n,\mu}^{B,q} f\right)(x) = \sum_{k=0}^{n} w_k(x)(B_q f)(x; x_k)$$

where $(B_q f)(x; x_k)$ is the Birkhoff interpolation polynomial which is the solution of Abel-Gontcharoff interpolation problem, that is

$$(B_q f)^{(j)}(x; x_k) = f^{(j)}(x_{k+j}), \quad k = \overline{0,n},\ j = \overline{0,q}.$$

For this problem the existence and the uniqueness of Birkhoff interpolation polynomial is guaranteed.

4. EXAMPLES AND GRAPHS

Let us consider the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = \sin 2\pi x$. In the sequel, for each class of example, we shall choose $\mu = 2$ and $\mu = 3$ and equispaced and Chebyshev nodes.

The example classes which we consider are the following:

- simple Shepard interpolation:

$$\left(S_{n,\mu}^{0} f\right)(x) = \sum_{k=0}^{n} w_k(x) f(x_k);$$

- Shepard-Taylor interpolation for $q = 1$:

$$\left(S_{n,\mu}^{1} f\right)(x) = \sum_{k=0}^{n} w_k(x)\left[f(x_k) + (x - x_k) f'(x_k)\right];$$

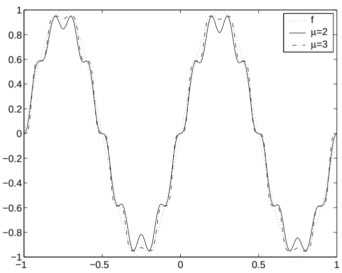
- Shepard-Lagrange interpolation for $q = 1$:

$$\left(S_{n,\mu}^{L,1}f\right) = \sum_{k=0}^{n} w_k(x)\left[\tfrac{x-x_{k+1}}{x_k-x_{k+1}}f(x_k) + \tfrac{x-x_k}{x_k-x_{k+1}}f(x_{k+1})\right].$$

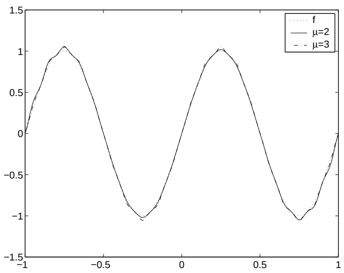
- Shepard-Birkhoff interpolation (Abel-Gontcharoff) for $q = 1$

$$\left(S_{n,\mu}^{L,1}f\right)(x) = \sum_{k=0}^{n} w_k(x)\left[f'(x_{k+1})(x-x_k) + f(x_k)\right].$$

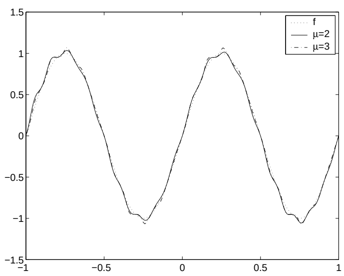
Figure 1 gives the graphs of various Shepard interpolants for equispaced nodes and the figure 2 gives the graph of the same interpolants for Cebyshev nodes. The graphs were generated using a MATLAB[2] toolbox, written by the authors (see [12]).



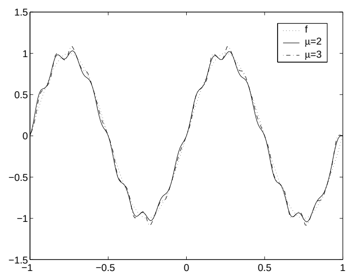
(a) Simple Shepard, equispaced nodes



(b) Shepard-Taylor, equispaced nodes



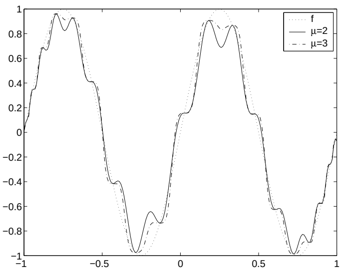
(c) Shepard-Lagrange, equispaced nodes



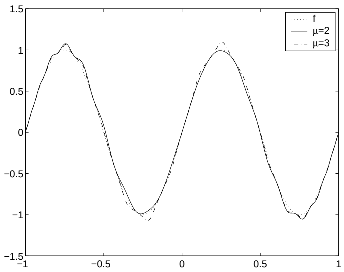
(d) Shepard-Birkhoff, equispaced nodes

Fig. 1. Various Shepard interpolants for equispaced nodes, $\mu = 2$ and $\mu = 3$

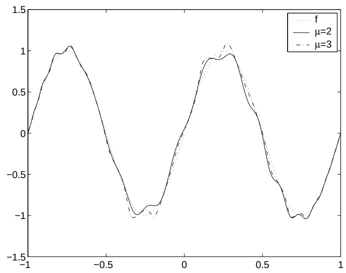
[2]MATLAB© is a trademark of MathWorks Inc, Natick, MA.



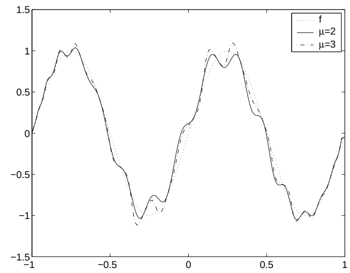
(a) Simple Shepard, Cebyshev nodes



(b) Shepard-Taylor, Cebyshev nodes



(c) Shepard-Lagrange, Cebyshev nodes



(d) Shepard-Birkhoff, Cebyshev nodes

Fig. 2. Various Shepard interpolants for Cebyshev nodes, $\mu = 2$ and $\mu = 3$

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