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BILEVEL TRANSPORTATION PROBLEMS

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In this paper we formulate the bilevel transportation problem of the cost-time type and of the cost-cost type, we propose a general algorithm for solving this problems and we also give two examples.

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1. INTRODUCTION

The bilevel (or two level) mathematical programming problem is an optimization problem with special constraints determined by, all or in part, another optimization problem. The bilevel mathematical programming problem is defined conventionally as follows:

$$\begin{aligned}&\text{Find } x^* \in \mathbb{R}^n \text{ such that } x^* \text{ solves}\\&\min F(x, y(x))\\&\text{subject to } G(x, y(x)) \leq 0,\end{aligned}$$

where $y(x)$ solves, for fixed x,

$$\begin{aligned}&\min f(x, y)\\&\text{subject to } g(x, y) \leq 0,\end{aligned}$$

where $G : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^r$, $F : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}$, $g : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^s$, and $f : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}$, are functions.

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2. BILEVEL TRANSPORTATION PROBLEM OF THE COST-TIME TYPE

Let $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ be positive integers so that $a_1 + \ldots + a_m = b_1 + \ldots + b_n$. Let $M = \{1, \ldots, m\}$, $P = \{1, \ldots, p\}$, $S = \{p+1, \ldots, n\}$,

$$\Lambda = \left\{ X = [x_{ij}] \in \mathbb{N}^{m \times p} \mid \sum_{i \in M} x_{ij} = b_j, \ \forall j \in P \text{ and } \sum_{j \in P} x_{ij} \le a_i, \ \forall i \in M \right\}.$$

If $X \in \Lambda$, then we put

$$U(X) = \left\{ Y(X) \in \mathbb{N}^{m \times (n-p)} \mid \sum_{i \in M} y_{ij}(X) = b_j, \ \forall\, j \in S, \right.$$
$$\left. \sum_{j \in S} y_{ij}(X) = a_i - \sum_{j \in P} x_{ij}, \ \forall\, i \in M \right\}.$$

The bilevel transportation problem of the cost-time type is defined as follows:

$$\begin{array}{ll} \text{Find} & X^* \in \mathbb{N}^{m \times p} \text{ such that } X^* \text{ solves} \\ \min & g(X) = \sum_{i \in M} \sum_{j \in P} c_{ij} x_{ij} + \sum_{i \in M} \sum_{j \in S} c_{ij} y^*_{ij}(X) \\ \text{subject} & \text{to} \\ & \sum_{i \in M} x_{ij} = b_j, \quad j \in P, \\ & \sum_{j \in P} x_{ij} \le a_i, \quad i \in M, \\ & \sum_{j \in P} x_{ij} + \sum_{j \in S} y^*_{ij}(X) = a_i, \quad i \in M \\ & x_{ij} \in \mathbb{N}, \quad i \in M, \ j \in P, \end{array}$$

where $Y^*(X) = [y^*_{ij}(x)] \in U(X)$ solves, for fixed $X \in \Lambda$,

$$\begin{array}{l} \min \max \{ t_{ij} \cdot sgn\, y_{ij}(X) \mid (i,j) \in \{1, ..., m\} \times \{p+1, ..., n\} \} \\ \text{subject to} \\ \qquad \sum_{j \in S} y_{ij} = a_i - \sum_{j \in P} x_{ij}, \quad i \in M \\ \qquad \sum_{i \in M} y_{ij} = b_j, \quad j \in S \\ \qquad y_{ij} \in \mathbb{N}, \quad i \in M, \ j \in S. \end{array}$$

In the following we denote by C the matrix of costs and by T the matrix of times:

$$C = \begin{pmatrix} c_{11} & ... & c_{1n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} t_{1,p+1} & ... & t_{1n} \\ ... & ... & ... \\ t_{m,p+1} & ... & t_{mn} \end{pmatrix}.$$

If we denote by $\alpha_i = \sum_{j \in P} x_{ij},\ i \in M,$ we have

$$0 \le \alpha_i \le a_i,\ i \in M, \quad \text{and} \quad \sum_{i \in M} \alpha_i = \sum_{j \in S} b_j.$$

Let be

$$H = \left\{ \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m \,|\, 0 \le \alpha_i \le a_i,\ \forall\ i \in M,\ \sum_{i \in M} \alpha_i = \sum_{j \in S} b_j \right\}.$$

Let the parametric transportation problem of the time type be

$$(PTP) \qquad t^*(\alpha) = \min \max\{t_{ij} \cdot sgn\, y_{ij} :\ i \in M,\ j \in S\}$$

subject to

$$\sum_{j \in S} y_{ij} = \alpha_i, \quad i \in M$$

$$\sum_{i \in M} y_{ij} = b_j, \quad j \in S$$

$$y_{ij} \in \mathbb{N}, \quad i \in M, \quad j \in S$$

where $\alpha = (\alpha_1, ..., \alpha_m) \in H$. If $\alpha^0 \in H$, then by $(PTP(\alpha^0))$ we denote the transportation problem of the time type which is obtained from (PTP) for $\alpha = \alpha^0$.

Solving the problem (PTP), we obtain a split of the set H into a finite number of subsets $H_1, \dots, H_q \subseteq \mathbb{N}^m$, such that $H_1 \cup \dots \cup H_q = H$, and for each $k \in \{1, \dots, q\}$, there are a real number T_k, with the property

$$t^*(\alpha) = T_k \text{ for all } \alpha \in H_k,$$

and a matrix, which we denote by $Y^k(\alpha)$,

$$Y^k(\alpha) = \begin{pmatrix} y^k_{1,p+1}(\alpha) & ... & y^k_{1n}(\alpha) \\ ... & ... & \\ y^k_{m,p+1}(\alpha) & ... & y^k_{mn}(\alpha) \end{pmatrix}$$

such that $Y^k(\alpha)$ is an optimal solution for the problem $(PTP(\alpha))$ for each $\alpha \in H_k$. Then a solution of the problem (PTP) is a function $h : H \to \mathbb{N}^{m \times (n-p)}$, given by

$$h(\alpha) \;=\; Y^k(\alpha), \quad \forall\, \alpha \in H_k,$$

for each $k \in \{1, ..., q\}$.

For each $k \in \{1, \ldots, q\}$, we solve the parametric transportation problem of the cost type

$$(TP_k) \qquad \min\left(F(X, Y^k(\alpha)) \;=\; \sum_{i\in M}\sum_{j\in P} c_{ij}x_{ij} \;+\; \sum_{i\in M}\sum_{j\in S} c_{ij}y^k_{ij}(\alpha)\right)$$

subject to

$$\sum_{i\in M} x_{ij} = b_j, \quad \forall\, j \in P$$

$$\sum_{j\in P} x_{ij} \le a_i, \quad \forall\, i \in M$$

$$\sum_{j\in P} x_{ij} \;+\; \sum_{j\in S} y^k_{ij}(\alpha) = a_i, \quad \forall\, i \in M$$

$$x_{ij} \in \mathbb{N},\; \forall i \in M,\; \forall j \in P,$$

where $\alpha \in H_k$.

For each $\alpha \in H_k$ we denote by $X^k(\alpha)$ an optimal solution of the problem $(TP_k(\alpha))$, obtained from (TP_k) when α is fixed.

Let $\alpha^* \in H_k$, such that

$$F(X^k(\alpha^*), Y^k(\alpha^*)) \;=\; \min\left\{F(X^k(\alpha), Y^k(\alpha)) \,|\, \alpha \in H_k\right\}.$$

Then, we call the matrix $X^k = X^k(\alpha^*)$, the best solution of the problem (TP_k).

The solution of the bilevel transportation problem (BTP) is that X^k for which we have

$$g(X^k) = \min\{g(X^l) :\; l \in \{1, \ldots, q\}\}.$$

In order to illustrate the above algorithm we conclude with the following numerical example.

Example Let us consider the bilevel cost-time transportation problem which has: $m = 3$, $p = 3$, $n = 4$, $a_1 = 80, a_2 = 25, a_3 = 45$, $b_1 = 75, b_2 = 40, b_3 = 15, b_4 = 20$, the times matrix : $T = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, and the costs matrix :

$$C = \begin{pmatrix} 3 & 2 & 5 & 1 \\ 1 & 3 & 2 & 4 \\ 5 & 2 & 4 & 3 \end{pmatrix}$$

The set H is

$$H = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \,|\, 0 \le \alpha_1 \le 80,\ 0 \le \alpha_2 \le 25,\ 0 \le \alpha_3 \le 45,\\ \alpha_1 + \alpha_2 + \alpha_3 = 20\}.$$

We have

$$H = H_1 \bigcup H_2 \bigcup H_3,$$

where

$$\begin{aligned} H_1 &= \{(0, 20, 0)\}, \\ H_2 &= \{(\alpha_1, \alpha_2, 0) \in \mathbb{N}^3 \,|\, 1 \le \alpha_1,\ \alpha_1 + \alpha_2 = 20\}, \\ H_3 &= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \,|\, \alpha_3 \ge 1, \alpha_1 + \alpha_2 + \alpha_3 = 20\}, \end{aligned}$$

Then

$$\begin{aligned} Y^1(\alpha) &= \begin{pmatrix} 0 \\ 20 \\ 2 \end{pmatrix}, \quad Y^2(\alpha) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix}, \quad Y^3(\alpha) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \\ X^1(\alpha) &= \begin{pmatrix} 70 & 10 & 0 \\ 5 & 0 & 0 \\ 0 & 30 & 15 \end{pmatrix}, \quad \text{for} \quad \alpha \in H_1, \\ X^2(\alpha) &= \begin{pmatrix} 70 - \alpha_1 & 10 & 0 \\ 25 - \alpha_2 & 0 & 0 \\ 0 & 30 & 15 \end{pmatrix}, \quad \text{for} \quad \alpha \in H_2, \\ X^3(\alpha) &= \begin{pmatrix} 70 - \alpha_1 - \alpha_3 & \alpha_3 + 10 & 0 \\ 5 + \alpha_1 + \alpha_3 & 0 & 0 \\ 0 & 30 - \alpha_3 & 15 \end{pmatrix}, \quad \text{for} \quad \alpha \in H_3, \\ &\min \{F(X^1(\alpha), Y^1(\alpha)) \,|\, \alpha \in H_1\} = 435 = F(X^1(0, 20, 0), Y^1(0, 20, 0)), \\ &\min \{F(X^2(\alpha), Y^2(\alpha)) \,|\, \alpha \in H_2\} = 335 = F(X^2(20, 0, 0), Y^2(20, 0, 0)), \\ &\min \{F(X^3(\alpha), Y^3(\alpha)) \,|\, \alpha \in H_3\} = 337 = F(X^3(19, 0, 1), Y^3(19, 0, 1)). \end{aligned}$$

Hence

$$X^1 = \begin{pmatrix} 70 & 10 & 0 \\ 5 & 0 & 0 \\ 0 & 30 & 15 \end{pmatrix}, \quad \text{and} \quad Y^1(X^1) = \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix},$$

$$X^2 = \begin{pmatrix} 50 & 10 & 0 \\ 25 & 0 & 0 \\ 0 & 30 & 15 \end{pmatrix}, \quad \text{and} \quad Y^2(X^2) = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix},$$

$$X^3 = \begin{pmatrix} 50 & 11 & 0 \\ 25 & 0 & 0 \\ 0 & 29 & 15 \end{pmatrix}, \quad \text{and} \quad Y^3(X^3) = \begin{pmatrix} 19 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that

$$\min \left\{ g(X^1),\, g(X^2),\, g(X^3) \right\} = 335 = g(X^2).$$

Hence the optimal solution for the bilevel transportation problem of the cost-time type is

$$X^* = X^2 = \begin{pmatrix} 50 & 10 & 0 \\ 25 & 0 & 0 \\ 0 & 30 & 15 \end{pmatrix}, \quad \text{and} \quad Y^*(X^*) = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}.$$

3. BILEVEL TRANSPORTATION PROBLEM OF THE COST-COST TYPE

Let $a_1, ..., a_m$, $b_1, ..., b_n$ be positive integers so that $a_1 + ... + a_m = b_1 + ... + b_n$. Let $M = \{1, ..., m\}$, $P = \{1, ..., p\}$, $S = \{p+1, ..., n\}$,

$$\Lambda = \left\{ X \in \mathbb{N}^{m \times p} \mid \sum_{i \in M} x_{ij} = b_j,\ \forall\, j \in P \quad \text{and} \quad \sum_{j \in P} x_{ij} \le a_i,\ \forall\, i \in M \right\}.$$

If $X \in \Lambda$, then we put

$$U(X) = \left\{ Y(X) \in \mathbb{N}^{m \times (n-p)} \mid \sum_{i \in M} y_{ij}(X) = b_j,\ \forall\, j \in S, \right.$$
$$\left. \sum_{j \in S} y_{ij}(X) = a_i - \sum_{j \in P} x_{ij},\ \forall\, i \in M \right\}.$$

The bilevel transportation problem of the cost-cost type is defined as follows:

$$\begin{array}{ll}
\text{Find} & X^* \in \mathbb{N}^{m\times p} \text{ such that } X^* \text{ solves} \\
\min & g(X) = \sum_{i\in M}\sum_{j\in P} c_{ij}x_{ij} + \sum_{i\in M}\sum_{j\in S} c_{ij}y^*_{ij}(X) \\
\text{subject} & \text{to} \\
& \sum_{i\in M} x_{ij} = b_j, \quad j \in P, \\
& \sum_{j\in P} x_{ij} \le a_i, \quad i \in M, \\
& \sum_{j\in P} x_{ij} + \sum_{j\in S} y^*_{ij}(X) = a_i, \quad i \in M \\
& x_{ij} \in \mathbb{N}, \quad i \in M,\ j \in P,
\end{array}$$

where $Y^*(X) \in U(X)$ solves, for fixed $X \in \Lambda$

$$\begin{array}{l}
\min \sum_{i=1}^{m}\sum_{j=p+1}^{n} d_{ij}y_{ij}(X) \\
\text{subject to} \\
\quad \sum_{j\in S} y_{ij} = a_i - \sum_{j\in P} x_{ij}, \quad i \in M \\
\quad \sum_{i\in M} y_{ij} = b_j, \quad j \in S \\
\quad y_{ij} \in \mathbb{N}, \quad i \in M,\ j \in S.
\end{array}$$

In the following we denote by C and D the matrices of costs:

$$C = \begin{pmatrix} c_{11} & ... & c_{1n} \\ ... & ... & ... \\ c_{m1} & ... & c_{mn} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_{1,p+1} & ... & d_{1n} \\ ... & ... & ... \\ d_{m,p+1} & ... & d_{mn} \end{pmatrix}.$$

Let be

$$H = \left\{ \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m \,|\, 0 \le \alpha_i \le a_i,\ \forall\ i \in M,\ \sum_{i\in M} \alpha_i = \sum_{j\in S} b_j \right\}.$$

Let the parametric transportation problem of the cost type be

$$(PCP) \qquad g^*(\alpha) = \min \sum_{i\in M}\sum_{j\in S} d_{ij}y_{ij}$$

subject to

$$\sum_{j\in S} y_{ij} = \alpha_i, \quad i \in M$$

$$\sum_{i\in M} y_{ij} = b_j, \quad j \in S$$

$$y_{ij} \in \mathbb{N}, \quad i \in M, \quad j \in S$$

where $\alpha = (\alpha_1, ..., \alpha_m) \in H$. If $\alpha^0 \in H$, then by $(PCP(\alpha^0))$ we denote the transportation problem of the cost type which is obtained from (PCP) for $\alpha = \alpha^0$.

Solving the problem (PCP), we obtain a split of the set H into a finite number of subsets $H_1, \ldots, H_q \subseteq \mathbb{N}^m$, such that $H_1 \cup \cdots \cup H_q = H$, and for each $k \in \{1, \ldots, q\}$, there are a real number d_k, with the property

$$g^*(\alpha) = d_k \text{ for all } \alpha \in H_k,$$

and a matrix, which we denote by $Y^k(\alpha)$,

$$Y^k(\alpha) = \begin{pmatrix} y^k_{1,p+1}(\alpha) & ... & y^k_{1n}(\alpha) \\ ... & ... & \\ y^k_{m,p+1}(\alpha) & ... & y^k_{mn}(\alpha) \end{pmatrix}$$

such that $Y^k(\alpha)$ is an optimal solution for the problem $(PCP(\alpha))$ for each $\alpha \in H_k$. Then a solution of the problem (PCP) is a function $h : H \to \mathbb{N}^{m\times(n-p)}$, given by

$$h(\alpha) = Y^k(\alpha), \quad \forall\, \alpha \in H_k,$$

for each $k \in \{1, ..., q\}$.

For each $k \in \{1, \ldots, q\}$, we solve the parametric transportation problem of the cost type

$$(CP_k) \qquad \min\left(F(X, Y^k(\alpha)) = \sum_{i\in M}\sum_{j\in P} c_{ij}x_{ij} + \sum_{i\in M}\sum_{j\in S} c_{ij}y^k_{ij}(\alpha)\right)$$

subject to

$$\sum_{i\in M} x_{ij} = b_j, \quad \forall\, j \in P$$

$$\sum_{j\in P} x_{ij} \le a_i, \quad \forall\, i \in M$$

$$\sum_{j\in P} x_{ij} + \sum_{j\in S} y^k_{ij}(\alpha) = a_i, \quad \forall\, i \in M$$

$$x_{ij} \in \mathbb{N},\ \forall i \in M,\ \forall j \in P,$$

where $\alpha \in H_k$.

For each $\alpha \in H_k$ we denote by $X^k(\alpha)$ an optimal solution of the problem $(CP_k(\alpha))$, obtained from (CP_k) when α is fixed.

Let $\alpha^* \in H_k$, such that

$$F(X^k(\alpha^*), Y^k(\alpha^*)) \;=\; \min \left\{ F(X^k(\alpha), Y^k(\alpha)) \,|\, \alpha \in H_k \right\}.$$

Then, we call the matrix $X^k = X^k(\alpha^*)$, the best solution of the problem (CP_k).

The solution of the bilevel transportation problem (BCP) is that X^k for which we have

$$g(X^k) = \min\{g(X^l) : \; l \in \{1, \ldots, q\}\}.$$

In order to illustrate the above algorithm we conclude with the following numerical example.

Example Let us consider the bilevel cost-cost transportation problem which has: $m = 2$, $p = 2$, $n = 4$, $a_1 = 80$, $a_2 = 70$, $b_1 = 20$, $b_2 = 10$, $b_3 = 90$, $b_4 = 30$, the first costs matrix : $D = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, and the second costs matrix :

$$C = \begin{pmatrix} 3 & 2 & 5 & 3 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$

The set H is

$$H \;=\; \left\{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \,|\, 0 \,\leq\, \alpha_1 \,\leq\, 80,\; 0 \,\leq\, \alpha_2 \,\leq\, 70,\; \alpha_1 \,+\, \alpha_2 \,=\, 120\right\}.$$

It is easy to see that

$$H \;=\; \left\{(\alpha_1, 120 - \alpha_1) \in \mathbb{N}^2 \,|50 \,\leq\, \alpha_1 \,\leq 80\right\}.$$

For each $\alpha = (\alpha_1, 120 - \alpha_1) \in H$ the transportation problem $(PCP(\alpha))$ has the optimal solution

$$Y^*(\alpha) \;=\; \begin{pmatrix} \alpha_1 - 30 & 30 \\ 120 - \alpha_1 & 0 \end{pmatrix}$$

Then, if $\alpha_1 \in H_1 \;=\; \{(\alpha_1, 120 - \alpha_1) \,|\, \alpha_1 \in [50, 70]\}$ we get

$$X^1(\alpha) \;=\; \begin{pmatrix} 70 - \alpha_1 & 10 \\ \alpha_1 - 50 & 0 \end{pmatrix},$$

and if $\alpha_1 \in H_2 \;=\; \{(\alpha_1, 120 - \alpha_1) \,|\, \alpha_1 \in [70, 80]\}$, we get

$$X^2(\alpha) \;=\; \begin{pmatrix} 0 & 80 - \alpha_1 \\ 20 & \alpha_1 - 70 \end{pmatrix}.$$

Then

$$\min \left\{F(X^1(\alpha), Y^*(\alpha)) \,|\, \alpha_1 \in [50, 70] \right\} \;=\; 430 \;=\; F(X^1(70, 50), Y^*(70, 50)),$$

and

$$\min\left\{F(X^2(\alpha), Y^*(\alpha)) \,|\, \alpha \in [70,80]\right\} = 430 = F(X^2(70,50), Y^*(70,50)).$$

Hence the optimal solution for the bilevel transportation problem of the cost-cost type is

$$X^* = \begin{pmatrix} 0 & 10 \\ 20 & 0 \end{pmatrix}, \quad \text{and} \quad Y^*(X^*) = \begin{pmatrix} 40 & 30 \\ 50 & 0 \end{pmatrix}.$$

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