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ON STABILITY AND QUASI-STABILITY OF A VECTOR LEXICOGRAPHIC QUADRATIC BOOLEAN PROGRAMMING PROBLEM*

V. A. EMELICHEV and YU. V. NIKULIN†

Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. We consider a vector Boolean programming problem with the linear-quadratic partial criteria. Formulas of radiuses of two types of stability, necessary and sufficient conditions of stability are found.

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This paper adjoins to the cycle of works [1]–[9], where different types of stability of a vector (multi-criterion) discrete lexicographic optimization problems were studied. In [1]–[4] a vector lexicographic problem on a system of subsets of a finite set with linear (MINSUM) partial criteria and some kinds of bottleneck (MINMAX) partial criteria is considered. Formulas for radiuses of three types of stability were found. The papers [6]–[8] are devoted to finding stability conditions and bounds of changing of input parameters in a vector integer linear programming problem. In [9] a regularization operator, that transforms any non-stable problem to some chain of stable problems, was found. Lower and upper attainable estimates for the stability radius of vector quadratic problem of consequent optimization were specified.

In this paper we consider vector Boolean programming problem with linear-quadratic partial criteria. It consists in finding the lexicographic set.

We study two types of stability of such problem. It is evident, that the stability (quasi-stability) of discrete problem is an equivalent of the famous property of upper (lower) semicontinity by Hausdorff of the optimal mapping, that determines correspondence between the vector criteria parameters and the lexicographic set. Formulas of radiuses of these types of stability, necessary and sufficient conditions of stability are found.

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†Mechanics–Mathematics Department, Belarusian State University, F. Scoryna av. 4, 220050 Minsk, Belarus, e-mail: eva@mmf.bsu.unibel.by.

Note, that in [10] a behavior of the Pareto set under independent perturbations of parameters in vector quadratic Boolean programming problem was studied.

1. BASE DEFINITIONS

Let m be the number of criteria, n be the number of elements, $A = (A_1, A_2, ..., A_m)$, $b = (b_1,b_2, ...,b_m)$, $m \in \mathbb{N}$, where any index $k \in N_m = \{1, 2, ..., m\}$ matrix $A_k \in \mathbb{R}^{n\times n}$, vector $b_k \in \mathbb{R}^n$, $n \in \mathbb{N}$, i.e. $A = [a_{ijk}] \in \mathbb{R}^{n\times n\times m}$, $b = [b_{ik}] \in \mathbb{R}^{n\times m}$. Here $\mathbb{N}$ ($\mathbb{R}$) is the set of natural (real) numbers.

Let $\mathbf{E}^n$ be the set of vertices of ort n-dimensional cube, i.e. $\mathbf{E}^n = \{0, 1\}^n$.

We assign a vector criterion

$$f(x) = (f_1(x), f_2(x), ..., f_m(x)) \longrightarrow \min_{x\in X}$$

on a set of Boolean vectors $X \subseteq \mathbf{E}^n$, $|X| > 1$. The partial criteria are the linear-quadratic functions

$$f_k(x) = \langle A_k x, x\rangle + \langle b_k, x\rangle \longrightarrow \min_{x\in X},\ k \in N_m,$$

where $\langle\cdot, \cdot\rangle$ is the scalar product of vectors, $x = (x_1, x_1, ..., x_n)^T$.

By changing the elements of pair (A, b), we obtain different vector criteria. Therefore, the pair (A, b) can be used for indexing the vector criterion $f(x)$ when the set of solutions X is fixed. The vector criterion is denoted by $f(x, A, b)$, and partial criterion is denoted by $f_k(x, A_k, b_k)$.

Further for any index $k \in N_m$ we will use notations

$$q_k(x, x', A_k, b_k) = f_k(x, A_k, b_k) - f_k(x', A_k, b_k).$$

The binary relation $\leq_s$ of lexicographic order is determined for a fixed permutations $s = (s_1, s_2, ..., s_m) \in S_m$ as follows:

$$x \leq_s x' \Longleftrightarrow (f(x, A, b) = f(x', A, b))\vee$$

$$(\exists j \in N_m\ \forall k \in N_{j-1}\ (q_{s_j}(x, x', A_{s_j}, b_{s_j}) < 0,\ \&\ q_{s_k}(x, x', A_{s_k}, b_{s_k}) = 0)),$$

where $N_0 = \emptyset$ (for $j = 1$).

Suppose S_m is the set of all $m!$ permutations of the numbers $1, 2, ..., m$.

We consider the problem of finding the lexicographic set $Z^m(A,b)$. It is a subset of the Pareto set and is defined as follows:

$$L^m(A,b) = \bigcup_{s \in S_m} L^m(A,b,s),$$

where

$$L^m(A,b,s) = \{x \in X : \ x \leq_s x' \ \forall x' \in X\}.$$

The elements of the set $L^m(A,b)$ are called lexicographic optima of the problem $Z^m(A,b)$. It is easy to see, that any lexicographic optimum belongs to the Pareto set

$$P^m(A,b) = \{x \in X : \ \pi(x,A,b) = \emptyset\},$$

where

$$\pi(x,A,b) = \{x' \in X \backslash \{x\} : q(x,x',A,b) \geq 0_{(m)}, q(x,x',A,b) \neq 0_{(m)}\},$$

$$q(x,x',A,b) = (q_1(x,x',A_1,b_1), q_2(x,x',A_2,b_2), ..., q_m(x,x',A_m,b_m)),$$

$$0_{(m)} = (0,0,...,0) \in \mathbb{R}^m.$$

We will give an equivalent definition of the lexicographic set $L^m(A,b,s)$:

$$L^m(A,b,s) = \{x \in X : \ \lambda(x) = \emptyset\},$$

where

$$\lambda(x) = \{x' \in X : \ x \prec_s x'\},$$

$$x \prec_s x' \Longleftrightarrow q_{s_i}(x,x',A_{s_i},b_{s_i}) < 0,$$

$$i = \min\{k \in N_m : \ q_{s_k}(x,x',A_{s_k},b_{s_k}) \neq 0\}.$$

Note that the set $L^m(A,b,s)$ may be obtained as a result of the solution of the single-criterion (scalar) problems sequence

$$L_k = arg\min\{f_{s_k}(x,A_{s_k},b_{s_k}) : \ x \in L_{k-1}\}, \ k \in N_m,$$

where $L_0 = X$. Thus, $L^m(A,b,s) = L_m$.

Our problem is the scalar quadratic Boolean programming problem and $L^1(A,b)$ is the set of optimal solutions for $m = 1$. The quadratic assignment problem and different optimization problems on graphs are represented in the scheme of the problem $L^1(A,b)$. It has many applications in electronics design: partitioning problem, covering problem, packing problem etc.

We assign the norm l_∞ for any number $p \in \mathbb{N}$

$$\| y \|_\infty = \max\{|y_i| : i \in N_p\}$$

in the space $\mathbb{R}^p$, and the norm l_1

$$\| y \|_1 = \sum_{i=1}^{p} |y_i|$$

in the space conjugate to $\mathbb{R}^p$.

The first one is called Chebyshev norm.

Under a matrix norm we understand the norm of vector, containing all the matrix elements.

Let $\varepsilon > 0$. As usually (see, e.g., [1-11]), we will perturb the parameters of vector criterion, i.e. the elements of pair (A, b) by adding to it a pair (A', b') from the set

$$\Omega(\varepsilon) = \{(A', b') \in \mathbb{R}^{n\times n\times m} \times \mathbb{R}^{n\times m} : \ ||A'||_\infty < \varepsilon, \ ||b'||_\infty < \varepsilon\},$$

where

$$A' = (A'_1, A'_2, ..., A'_m), \ b' = (b'_1, b'_2, ..., b'_m),$$

$$A'_k \in \mathbb{R}^{n\times n}, \ b'_k \in \mathbb{R}^n, \ k \in N_m.$$

The problem $Z^m(A + A', b + b')$, where $(A', b') \in \Omega(\varepsilon)$,

$$A + A' = (A_1 + A'_1, A_2 + A'_2, ..., A_m + A'_m),$$

$$b + b' = (b_1 + b'_1, b_2 + b'_2, ..., b_m + b'_m),$$

obtained from the initial problem $Z^m(A, b)$ by addition of corresponding vectors and matrices, is called perturbed. The pair (A', b') is called perturbing.

According to [1]–[9], the problem $Z^m(A, b)$ is called

– stable, if

$$\exists \varepsilon > 0 \ \ \forall (A', b') \in \Omega(\varepsilon) \ (L^m(A + A', b + b') \subseteq L^m(A, b)),$$

– quasi-stable, if

$$\exists \varepsilon > 0 \ \ \forall (A', b') \in \Omega(\varepsilon) \ (L^m(A, b) \subseteq L^m(A + A', b + b')).$$

It's evident, that the stability (quasi-stability) of discrete problem $Z^m(A, b)$ is an analog of the famous property (see, e.g., [12, 13]) of upper (lower) semicontinity by Hausdorff in the point $(A, b) \in \mathbb{R}^{n\times n\times m} \times \mathbb{R}^{n\times m}$ of the optimal mapping

$$L^m : \ \mathbb{R}^{n\times n\times m} \times \mathbb{R}^{n\times m} \longrightarrow 2^{\mathbf{E}},$$

i.e. the many-valued mapping that defines the choice function.

2. PROPERTIES AND LEMMA

Taken place the next evident properties.

PROPERTY 1. *A solution x is lexicographic optimum of the problem* $Z^m(A,b)$, *i.e.* $x \in L^m(A,b)$, *if there exists an index* $k \in N_m$ *such that the inequality* $q_k(x, x', A_k, b_k) < 0$ *is hold for any solution* $x' \in X \backslash \{x\}$.

It is easy to see, that the inverse statement is false in general.

PROPERTY 2. $x \notin L^m(A,b)$, *if for any index* $k \in N_m$ *there exists a solution* $x' \in X \backslash \{x\}$ *such that the inequality* $q_k(x, x', A_k, b_k) > 0$ *is true.*

The following statements are true for any vectors $x, x' \in \mathbf{E}^n$, $c \in \mathbb{R}^n$:

(1) $$|\langle c, x\rangle| \leq \| c \|_\infty \cdot \| x \|_1,$$

(2) $$\| x - x' \|_1 = \| x \|_1 + \| x' \|_1 - 2\langle x, x'\rangle,$$

(3) $$\| \tilde{x} \|_1 = \| x \|_1^2,$$

(4) $$\langle \tilde{x}, \tilde{x}'\rangle = \langle x, x'\rangle^2,$$

where $\tilde{x} = (x_1x_1, x_1x_2, ..., x_nx_{n-1}, x_nx_n)$, $\tilde{x}' = (x'_1x'_1, x'_1x'_2, ..., x'_nx'_{n-1}, x'_nx'_n)$.

Note, that the left-hand side of equality (2) is the Hamming distance between Boolean vectors x and x'. It is easy to prove equality (2) using the induction (on the number n).

LEMMA 1. *Let the inequality*

(5) $$q_k(x, x', A_k, b_k) + ||A'_k||_\infty(||x||_1^2 + ||x'||_1^2 - 2\langle x, x'\rangle^2) + ||b'_k||_\infty \cdot ||x - x'||_1 < 0,$$

holds for any index $k \in N_m$, *where* $x, x' \in X$, $A_k, A'_k \in \mathbb{R}^{n\times n}$, $b_k, b'_k \in \mathbb{R}^n$.

Then the inequality

$$q_k(x, x', A_k + A'_k, b_k + b'_k) < 0.$$

is true

Really, consequently applying statements (1)–(4) and lemma condition, we get

$$\begin{aligned}
&q_k(x, x', A_k + A'_k, b_k + b'_k) \leq \\
&\leq q_k(x, x', A_k, b_k) + |\langle A'_k x, x\rangle - \langle A'_k x', x'\rangle| + |\langle b'_k, x - x'\rangle| \\
&\leq q_k(x, x', A_k, b_k) + ||A'_k||_\infty \sum_{i=1}^n \sum_{j=1}^n |x_ix_j - x'_ix'_j| + ||b'_k||_\infty \cdot ||x - x'||_1 \\
&= q_k(x, x', A_k, b_k) + ||A'_k||_\infty(\| \tilde{x} \|_1 + \| \tilde{x}' \|_1 - 2\langle \tilde{x}, \tilde{x}'\rangle) + ||b'_k||_\infty \cdot ||x - x'||_1 \\
&= q_k(x, x', A_k, b_k) + ||A'_k||_\infty(||x||_1^2 + ||x'||_1^2 - 2\langle x, x'\rangle^2) + ||b'_k||_\infty \cdot ||x - x'||_1 \\
&< 0.
\end{aligned}$$

3. THE STABILITY RADIUS

The number (see [1], [2])

$$\rho_1^m(A,b) = \begin{cases} \sup \Theta_1(A,b), & \text{if } \Theta_1(A,b) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Theta_1(A,b) = \{\varepsilon > 0 : \ \forall (A',b') \in \Omega(\varepsilon) \ (L^m(A+A', b+b') \subseteq L^m(A,b)) \ \},$$

is called the stability radius of the problem $Z^m(A,b)$.

Thus, the stability radius of the problem $Z^m(A,b)$ is the limit of independent perturbations of elements of (A,b) such that new lexicographic optima do not appear.

It is clear, that the stability radius is infinite as $X = L^m(A,b)$. Therefore we will exclude this case from the consideration. We call the problem $Z^m(A,b)$ non-trivial, if $\bar{L}^m(A,b) = X \backslash L^m(A,b) \neq \emptyset$.

THEOREM 1. *Let the problem $Z^m(A,b), m \geq 1$, be non-trivial. Then the stability radius is expressed by the formula*

$$\rho_1^m(A,b) = \min_{x \in \bar{L}^m(A,b)} \min_{k \in N_m} \max_{x' \in X \backslash \{x\}} \frac{q_k(x,x',A_k,b_k)}{||x||_1^2 + ||x'||_1^2 + ||x-x'||_1 - 2\langle x,x' \rangle^2}. \tag{6}$$

Proof. Let φ denote the right part of equality (6). Then $\varphi \geq 0$. First let us prove the inequality

$$\rho_1^m(A,b) \geq \varphi. \tag{7}$$

There is nothing to prove for $\varphi = 0$.

Let $\varphi > 0$. According to the definition of φ, for any solution $x \in \bar{L}^m(A,b)$ (since the problem is non-trivial, such solution exists) and for any index $k \in N_m$, there exists a solution $x' \in X \backslash \{x\}$ such that the inequality

$$q_k(x,x',A_k,b_k) \geq \varphi(||x||_1^2 + ||x'||_1^2 + ||x-x'||_1 - 2\langle x,x' \rangle^2)$$

is true. Hence, using inequalities $||A'_k||_\infty < \varphi, \ ||b'_k||_\infty < \varphi$ and the lemma, we conclude, that for any perturbing pair $(A',b') \in \Omega(\varphi)$ and any index $k \in N_m$ the inequality

$$q_k(x', x, A + A', b + b') < 0$$

is true. So

$$q_k(x, x', A + A', b + b') > 0.$$

Hence, according to property 2, we get that the solution x does not belong to the lexicographic set of the perturbed problem $Z^m(A + A', b + b')$, $(A', b') \in \Omega(\varphi)$.

Thus, for any perturbing pair $(A', b') \in \Omega(\varphi)$ it follows that

$$L^m(A + A', b + b') \subseteq L^m(A, b).$$

Hence, estimate (7) holds.

Now let us prove, that $\rho_1^m(A, b) \leq \varphi$.

According to the definition of $\varphi \geq 0$, there exists a solution $x \in \bar{L}^m(A, b)$ and an index $p = p(x) \in N_m$ such that the inequality

$$\varphi(||x||_1^2 + ||x'||_1^2 + ||x - x'||_1 - 2\langle x, x'\rangle^2) \geq q_p(x, x', A_p, b_p) \tag{8}$$

holds for any solution $x' \in X\backslash\{x\}$. Consider the perturbing pair (A', b'), where $A' = (A'_1, A'_2, ..., A'_m)$, $b' = (b'_1, b'_2, ..., b'_m)$. The elements of matrix $A' = [a'_{ijk}]_{n\times n\times m}$ and the elements of vector $b' = [b'_{ik}]_{n\times m}$ are determined by setting

$$a'_{ijk} = \begin{cases} \alpha, & \text{if } k = p,\ x_i x_j = 0, \\ -\alpha, & \text{if } k = p,\ x_i x_j = 1, \\ 0, & \text{if } k \neq p,\ (i, j) \in N_n \times N_n, \end{cases} \qquad b'_{ik} = \begin{cases} \alpha, & \text{if } x_i = 0, \\ -\alpha, & \text{if } x_i = 1. \\ 0, & \text{if } k \neq p,\ i \in N_n, \end{cases}$$

Here $\varphi < \alpha < \varepsilon$. Then, $(A', b') \in \Omega(\varepsilon)$. Using (8) we deduce

$$q_p(x, x', A_p + A'_p, b_p + b'_p) =$$

$$= q_p(x, x', A_p, b_p) - \alpha(||x'||_1^2 + ||x||_1^2 + ||x - x'||_1 - 2\langle x, x'\rangle^2) < 0.$$

Combining it with property 1, we have $x \in L^m(A + A', b + b')$. Thus, for any number $\varepsilon > \varphi$ there exists a perturbing pair $(A', b') \in \Omega(\varepsilon)$ such that

$$L^m(A + A', b + b') \nsubseteq L^m(A, b).$$

Hence, for any number $\varepsilon > \varphi$ the inequality $\rho_1^m(A, b) < \varepsilon$ is true, i.e. $\rho_1^m(A, b) \leq \varphi$.

Theorem 1 is proved. □

Let us introduce the set of weak optima of the problem $Z^m(A,b)$

$$S_1^m(A,b) = \{x \in X : \ \exists k = k(x) \in N_m \ \forall x' \in X \backslash \{x\} \ (q_k(x,x',A_k,b_k) \leq 0)\}.$$

Applying property 2, we get

$$L^m(A,b) \subseteq S_1^m(A,b). \tag{9}$$

Since the problem $Z^m(A,b)$ is stable, iff $\rho_1^m(A,b) > 0$, from theorem 1 we get the following corollary.

Corollary 1. *The non-trivial problem $Z^m(A,b)$, $m \geq 1$, is stable, the equality $L^m(A,b) = S_1^m(A,b)$ is true.*

Proof. Necessity. Let the non-trivial problem $Z^m(A,b)$ be stable. Then, according to theorem 1, the number φ (the right part of formula (6)) is positive. Therefore for any solution $x \in \bar{L}^m(A,b)$ and for any index $k \in N_m$ there exists a solution $x' \in X\backslash\{x\}$ such that the inequality $q_k(x,x',A_k,b_k) > 0$ holds. Hence, according to the definition of the set of weak optima $S_1^m(A,b)$, we get

$$\bar{L}^m(A,b) \cap S_1^m(A,b) = \emptyset,$$

i.e. $S_1^m(A,b) \subseteq L^m(A,b)$. Hence, applying (9), we have $S_1^m(A,b) = L^m(A,b)$.

Sufficiency. Let $S_1^m(A,b) = L^m(A,b)$. Then, according to the definition of $S_1^m(A,b)$ for any solution $x \in \bar{L}^m(A,b) = \bar{S}_1^m(A,b)$ and any index $k \in N_m$ there exists a solution $x' \in X\backslash\{x\}$ such that $q_k(x,x',A_k,b_k) > 0$. Therefore $\varphi > 0$. Hence, by theorem 1, the problem $Z^m(A,b)$ is stable.

Corollary 1 is proved. □

We conclude from corollary 1, that any single-criterion problem $Z^1(A,b)$ is stable.

4. THE QUASI-STABILITY RADIUS

The number (see [2], [4]–[7])

$$\rho_2^m(A,b) = \begin{cases} \sup\Theta_2, & \text{if } \Theta_2 \neq \emptyset, \\ 0, & \text{if } \Theta_2 = \emptyset, \end{cases}$$

is called the quasi-stability radius of the problem $Z^m(A,b)$, where $\Theta_2 = \{\varepsilon > 0 : \ \forall (A',b') \in \Omega(\varepsilon) \ (L^m(A,b) \subseteq L^m(A+A',b+b'))\}$.

In other words, the quasi-stability radius is the limit of independent perturbations of elements of (A, b) such that all initial lexicographic optima preserve optimality in any perturbed problem. New optima may arise.

THEOREM 2. *The quasi-stability radius of the problem $Z^m(A, b)$, $m \geq 1$ is expressed by the formula*

(10)
$$\rho_2^m(A, b) = \min_{x' \in L^m(A,b)} \max_{k \in N_m} \min_{x \in X \backslash \{x'\}} \frac{q_k(x, x', A_k, b_k)}{||x||_1^2 + ||x'||_1^2 + ||x - x'||_1 - 2\langle x, x' \rangle^2}.$$

Proof. Let ψ denote the right part of (10). It is clear, that $\psi \geq 0$. First let us prove the inequality

$$\rho_2^m(A, b) \geq \psi. \tag{11}$$

There is nothing to prove for $\psi = 0$.

Let $\psi > 0$. Then according to the definition of ψ for any solution $x' \in L^m(A, b)$ there exists an index $p \in N_m$, such that the inequality

$$q_p(x, x', A_p, b_p) \geq \psi(||x||_1^2 + ||x'||_1^2 + ||x - x'||_1 - 2\langle x, x' \rangle^2)$$

is true for any solution $x \in X \backslash \{x'\}$. Applying $||A_p'||_\infty < \psi$, $||b_p'||_\infty < \psi$ and using lemma we conclude that for any perturbing pair $(A', b') \in \Omega(\psi)$ the inequality $q_p(x', x, A + A', b + b') < 0$ is true. So $q_p(x, x', A + A', b + b') > 0$.

Therefore, according to property 1, it follows that a solution x' belongs to the lexicographic set of the perturbed problem $Z^m(A + A', b + b')$, $(A', b') \in \Omega(\psi)$.

Thus, for any perturbing pair $(A', b') \in \Omega(\psi)$ we obtain

$$L^m(A, b) \subseteq L^m(A + A', b + b').$$

Hence, the estimate (11) holds.

Now let us prove, that $\rho_2^m(A, b) \leq \psi$.

According to the definition of $\psi \geq 0$, there exist a solution $x' \in L^m(A, b)$ such that for any index $k \in N_m$ there exists a solution $x \in X \backslash \{x'\}$ such that the inequality

$$\psi(||x||_1^2 + ||x'||_1^2 + ||x - x'||_1 - 2\langle x, x' \rangle^2) \geq q_k(x, x', A_k, b_k) \tag{12}$$

is true.

Consider the perturbing pair (A', b'), where$A' = (A_1', A_2', ..., A_m')$, $b' = (b_1', b_2', ..., b_m')$, the matrix $A' = [a_{ijk}']_{n \times n \times m}$ and the vector $b' = [b_{ik}']_{n \times m}$ are determined for any index $k \in N_m$ by setting:

$$a'_{ijk} = \begin{cases} -\alpha, & \text{if } x'_i x'_j = 0, \\ \alpha, & \text{if } x'_i x'_j = 1, \end{cases} \quad b'_{ik} = \begin{cases} -\alpha, & \text{if } x'_i = 0, \\ \alpha, & \text{if } x'_i = 1, \end{cases}$$

where $\psi < \alpha < \varepsilon$. Thus, $(A', b') \in \Omega(\varepsilon)$. Hence, combining it with (12), we deduce

$$q_k(x, x', A_k + A'_k, b_k + b'_k) =$$

$$= q_k(x, x', A_k, b_k) - \alpha(||x'||_1^2 + ||x||_1^2 + ||x - x'||_1 - 2\langle x, x' \rangle^2) < 0$$

for any index $k \in N_m$. Therefore, by property (2), $x \notin L^m(A+A', b+b')$. Thus, for any number $\varepsilon > \psi$ the inequality $\rho_2^m(A, b) < \varepsilon$ holds. So $\rho_2^m(A, b) \leq \psi$.

Theorem 2 is proved. □

Let us introduce the set of regular optima of the problem $Z^m(A, b)$:

$$S_2^m(A, b) = \{x' \in X : \ \exists k = k(x') \in N_m \ \forall x \in X \backslash \{x'\} \ (q_k(x, x', A_k, b_k) > 0.)$$

By property 1, it is easy to see

$$S_2^m(A, b) \subseteq L^m(A, b). \tag{13}$$

Corollary 2. *The vector problem $Z^m(A, b)$, $m \geq 1$, is quasi-stable, iff*

$$L^m(A, b) = S_2^m(A, b).$$

Proof. Necessity. Let the problem $Z^m(A, b)$ be quasi-stable. Then, according to theorem 2, the number ψ is positive. It follows that for any solution $x' \in L^m(A, b)$ there exists index $k \in N_m$, such that the inequality $q_k(x, x', A_k, b_k) > 0$ is true for any solution $x \in X \backslash \{x'\}$. According to the definition of the set $S_2^m(A, b)$, we get $x' \in S_2^m(A, b)$. Thus, $L^m(A, b) \subseteq S_2^m(A, b)$. Hence, considering (13) we have $S_2^m(A, b) = L^m(A, b)$.

Sufficiency. Let $S_2^m(A, b) = L^m(A, b)$. Then, by the definition of the set of regular optima $S_2^m(A, b)$ for any solution $x' \in L^m(A, b)$ there exists an index $k \in N_m$ such that the inequality $q_k(x, x', A_k, b_k) > 0$ holds for any solution $x \in X \backslash \{x'\}$. Therefore $\psi > 0$. Hence, by theorem 2, the problem $Z^m(A, b)$ is quasi-stable.

Corollary 2 is proved. □

We conclude the following results, from corollary 2.

Corollary 3. *Scalar problem $Z^1(A,b)$ is quasi-stable, iff it has a unique optimal solution.*

Corollary 4. *The problem $Z^m(A,b)$ is quasi-stable, iff $|L^m(A,b)| \leq m$.*

At the end of this paper we give an example of matrices $A^{(1)},\ A^{(2)},\ A^{(3)}$ and vectors $b^{(1)},\ b^{(2)},\ b^{(3)}$, such that the following statements are true

$$\rho_1^m(A^{(1)},b^{(1)}) < \rho_2^m(A^{(1)},b^{(1)}),$$

$$\rho_1^m(A^{(2)},b^{(2)}) > \rho_2^m(A^{(2)},b^{(2)}),$$

$$\rho_1^m(A^{(3)},b^{(3)}) = \rho_2^m(A^{(3)},b^{(3)}).$$

Let $n=m=2,\ X=\{x_1,x_2,x_3\},\ x_1=(1,1),\ x_2=(0,1)\ x_3=(1,0),$

$$A^{(1)} = (A_1^{(1)},A_2^{(1)}),\ b^{(1)} = (b_1^{(1)},b_2^{(1)}),$$

$$A^{(2)} = (A_1^{(2)},A_2^{(2)}),\ b^{(2)} = (b_1^{(2)},b_2^{(2)}),$$

$$A^{(3)} = (A_1^{(3)},A_2^{(3)}),\ b^{(3)} = (b_1^{(3)},b_2^{(3)}).$$

$$A_1^{(1)} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},\ A_2^{(1)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ b_1^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ b_2^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_1^{(2)} = \begin{pmatrix} 4 & 0 \\ -2 & 0 \end{pmatrix},\ A_2^{(2)} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix},\ b_1^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ b_2^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_1^{(3)} = \begin{pmatrix} 4 & 0 \\ -2 & 0 \end{pmatrix},\ A_2^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix},\ b_1^{(3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ b_2^{(3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have

$$f(x_1,A^{(1)},b^{(1)}) = (0,0),\ f(x_2,A^{(1)},b^{(1)}) = (0,1),\ f(x_3,A^{(1)},b^{(1)}) = (1,1),$$
$$f(x_1,A^{(2)},b^{(2)}) = (2,4),\ f(x_2,A^{(2)},b^{(2)}) = (0,1),\ f(x_3,A^{(2)},b^{(2)}) = (4,0),$$
$$f(x_1,A^{(3)},b^{(3)}) = (2,4),\ f(x_2,A^{(3)},b^{(3)}) = (0,4),\ f(x_3,A^{(3)},b^{(3)}) = (4,0).$$

Thus, applying theorems 1 and theorem 2, we obtain

$$\rho_1^2(A^{(1)},b^{(1)}) = 0,\ \rho_2^2(A^{(1)},b^{(1)}) = \tfrac{1}{2}, \tag{14}$$

$$\rho_1^2(A^{(2)},b^{(2)}) = 1,\ \rho_2^2(A^{(2)},b^{(2)}) = \tfrac{1}{2}, \tag{15}$$

$$\rho_1^2(A^{(3)},b^{(3)}) = 1,\ \rho_2^2(A^{(3)},b^{(3)}) = 1. \tag{16}$$

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