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A VORONOVSKAYA–TYPE THEOREM

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. We give an asymptotic estimation for some sequences of divided differences. We use this estimation to obtain a Voronovskaya–type formula involving linear positive operators.

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1. INTRODUCTION AND NOTATIONS

Consider the points $x_0 < x_1 < \ldots < x_n$ on the real axis and let $f\colon [x_0, x_n] \to \mathbb{R}$ be an arbitrary function. Denote by $[x_0, \ldots, x_n; f]$ the divided difference of the function f on the knots $x_0, \ldots, x_n$, usually defined by

$$[x_0, \ldots, x_n; f] := \sum_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}.$$

Consider the polynomial functions $e_i\colon \mathbb{R} \to \mathbb{R}$, $e_i(x) = x^i$, $i = 0, 1, \ldots$. It is known that $[x_0, \ldots, x_n; e_i] = 0$, $i = 0, \ldots, n-1$, $[x_0, \ldots, x_n; e_n] = 1$. The problem was to calculate $A_k := [x_0, \ldots, x_n; e_{n+k}]$, $k = 1, 2, \ldots$. In [6] Tiberiu Popoviciu uses the identity

$$\left[x_0, \ldots, x_n; \tfrac{1}{x - \cdot}\right] = \frac{1}{(x - x_0)\ldots(x - x_n)},$$

to prove the following formula

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$$A_k = \sum_{\substack{0\le i_0,\dots,i_n\le k\\ i_0+\dots+i_n=k}} x_0^{i_0}\cdots x_n^{i_n}.$$

This formula was rediscovered in 1981 by E. Neuman [3]. It does not look much “friendlier” than the initial one,

$$A_k = \sum_{i=0}^{n} \frac{x_i^{n+k}}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}.$$

Therefore, in [7], it is suggested that a recurrence formula might be more useful. We shall use such a formula in order to give an asymptotic estimation for A_k under some supplementary assumptions on the knots (see Theorem 1).

Consider now a triangular matrix of nodes $(x_{n,k})$, $n = 0, 1, \dots; k = 0, \dots, n$,

$$-1 \le x_{n,0} < x_{n,1} < \dots < x_{n,n} \le 1, \quad n = 0, 1, \dots \tag{1}$$

satisfying the conditions:

$$x_{n,n-i} = -x_{n,i}, \quad i = 0, \dots, n,\ n = 0, 1, \dots. \tag{2}$$

Let $a > 0$. For $n \ge 1$ consider the operator $L_n : C[-a-1, a+1] \to C[-a, a]$,

$$L_n f(x) := n![x + x_{n,0}, \dots, x + x_{n,n}; F_n],$$

where $f \in C[-a-1, a+1]$, $x \in [-a, a]$, $F_n \in C^n[-a-1, a+1]$, $F_n^{(n)} = f$.

The L_n are positive linear operators of probabilistic type and Bernstein–Schnabl type operators.

For particular choices of the matrix $(x_{n,k})$ various inequalities involving $L_n f$ have been studied in [4], [5], [8], [12]. If $x_{n,i} = -1 + \frac{2i}{n}$, $i = 0, \dots, n$, we have also [10]

$$L_n f(x) = 2^{-n} \int_{x-1}^{x+1} \cdots \int_{x-1}^{x+1} f\left(\tfrac{t_1+\dots+t_n}{n}\right) dt_1 \dots dt_n. \tag{3}$$

Using the L_n operator notation, [7] gives

$$\left| L_n f(0) - \sum_{i=0}^{k-1} \frac{L_n e_{2i}(0)}{(2i)!} f^{(2i)}(0) \right| \le \frac{L_n e_{2k}(0)}{(2k)!} \|f^{(2k)}\|_{[-1,1]}. \tag{4}$$

for all $f \in C^{2k}[-a-1, a+1]$, where $\|\cdot\|_{[-1,1]}$ denotes the uniform norm on $C[-1, 1]$. As positive operators, L_n have been studied in [9], [10].

They verify:

$$f \text{ convex } \Longrightarrow L_n f \ge L_{n+1} f \ge f,$$

$$\|L_n f - f\| \le 2\,\omega\left(f, \tfrac{1}{\sqrt{3n}}\right).$$

We have: $Le_0 = e_0$, $Le_1 = e_1$, $L(e_1 - x\,e_0)^2(x) = \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} x_{n,i}^2$.

For equidistant knots $x_{n,i} = -1 + \frac{2i}{n}$, $i = 0, \ldots, n$, we obtain

$$L(e_1 - x\,e_0)^2(x) = \tfrac{1}{3n},$$

hence, using [1, Corollary 4.12], we can prove now that

$$\|L_n f - f\| \le 2.25\,\omega_2\left(f, \tfrac{1}{\sqrt{3n}}\right).$$

Our aim is to give a more refined analysis of the convergence behaviour of the operators L_n. This is accomplished in Theorem 2.

2. MAIN RESULTS

Theorem 1. *If the triangular matrix* $(x_{n,k})$ *satisfies the relations* (1), (2) *and*

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n} x_{n,i}^2 = 2\lambda \in \mathbb{R} \tag{5}$$

then, for all $k \in \mathbf{N}$, *the following equality is fulfilled*

$$\lim_{n\to\infty} n^{-k}[x_{n,0}, \ldots, x_{n,n}; e_{n+2k}] = \tfrac{\lambda^k}{k!}. \tag{6}$$

The previous relation can be written in the form

$$\lim_{n\to\infty} n^k L_n e_{2k}(0) = \tfrac{\lambda^k}{k!}\,(2k)!. \tag{7}$$

Theorem 2. *If* $f \in C^{2k}[-a-1, a+1]$, *then, for every matrix* $(x_{n,k})$ *satisfying the conditions* (1), (2) *and* (5), *the following Voronovskaya–type relation holds true:*

$$\lim_{n\to\infty} n^k \left(L_n f(x) - \sum_{i=0}^{k-1} \tfrac{L_n e_{2i}(0)}{(2i)!}\, f^{(2i)}(x) \right) = \tfrac{\lambda^k}{k!} f^{(2k)}(x), \tag{8}$$

uniformly for $x \in [-a, a]$.

3. PROOF OF THEOREM 1.

Consider the polynomial function $(e_1-x_{n,0})\dots(e_1-x_{n,n})$, which we write as $e_{n+1}-C_{n,1}e_n-\cdots-C_{n,n}e_1-C_{n,n+1}e_0$. Consider also the sums $S_{n,p}:=\sum_{i=0}^{n}x_{n,i}^p$, $p=1,2,\dots$. Using (2) it is obvious that

$$S_{n,p}=0,\qquad \text{for odd } p. \tag{9}$$

Using (1) it can be easily shown that

$$\lim_{n\to\infty}\frac{S_{n,p}}{n^k}=0,\quad p=1,2,\dots;\quad k>1. \tag{10}$$

We write the relation (5) in the form

$$\lim_{n\to\infty}\frac{S_{n,2}}{n}=2\lambda. \tag{11}$$

The coefficients

$$C_{n,p}=(-1)^{p+1}\sum_{0\le i_1<\dots<i_p\le n}x_{n,i_1}\cdots x_{n,i_p},$$

can be computed by using Newton's formulas:

$$\begin{aligned} C_{n,1}&=S_{n,1}\\ C_{n,p}&=\frac{1}{p}\left(S_{n,p}-\sum_{i=1}^{p-1}S_{n,i}C_{n,p-i}\right),\quad p=2,\dots,n+1. \end{aligned}$$

By considering (9) it follows that:

$$C_{n,p}=0,\quad \text{for odd } p,$$

and

$$\begin{aligned} C_{n,2}&=\tfrac{1}{2}S_{n,2}\\ C_{n,2k}&=\frac{1}{2k}\left(S_{n,2k}-\sum_{i=1}^{k-1}S_{n,2i}C_{n,2(k-i)}\right),\quad k=1,\dots,\lfloor (n+1)/2\rfloor. \end{aligned} \tag{12}$$

We define γ_k as

$$\gamma_k:=\lim_{n\to\infty}\frac{C_{n,2k}}{n^k},\quad k=1,2,\dots.$$

We have

$$\gamma_1=\lim_{n\to\infty}\frac{C_{n,2}}{n}=\lim_{n\to\infty}\frac{S_{n,2}}{2n}=\lambda,$$

and, from (10) and (12), it follows that

$$\gamma_k = -\tfrac{\lambda}{k}\gamma_{k-1}, \quad k \geq 2$$

hence

$$\gamma_k = \frac{(-1)^{k+1}\lambda^k}{k!}, \quad k \geq 1. \tag{13}$$

Using the divided difference functional, define the numbers:

$$A_{n,j} := [x_{n,0}, \dots, x_{n,n}; e_{n+j}], \quad j = -n, -n+1, \dots.$$

It is well known that

$$A_{n,j} = \begin{cases} 0, & \text{if } j = -n, \dots, -1, \\ 1, & \text{if } j = 0. \end{cases} \tag{14}$$

In order to calculate $A_{n,j}$ for $j \geq 1$, observe that

$$[x_{n,0}, \dots, x_{n,n}; e_{j-1}(e_1 - x_{n,0}) \dots (e_1 - x_{n,n})] = 0,$$

that is

$$[x_{n,0}, \dots, x_{n,n}; e_{n+j} - C_{n,1}e_{n+j-1} - \dots - C_{n,n+1}e_{j-1}] = 0.$$

As a consequence we have

$$A_{n,j} = \sum_{i=1}^{n+1} C_{n,i}A_{n,j-i}, \quad j = 1, 2, \dots$$

and using (14), we find that

$$A_{n,j} = \sum_{i=1}^{j} C_{n,i}A_{n,j-i}, \quad j = 1, \dots, n+1. \tag{15}$$

Using

$$\begin{array}{rcl} A_{n,0} & = & 1 \\ A_{n,1} & = & C_{n,1}A_{n,0} = 0 \end{array}$$

in (15), it can be deduced that

$$A_{n,p} = 0, \qquad \text{for odd } p, \tag{16}$$

and hence

$$A_{n,2k} = \sum_{i=1}^{k} C_{n,2i}A_{n,2(k-i)}, \quad 1 \leq k \leq \tfrac{n+1}{2}. \tag{17}$$

By defining

$$B_k := \lim_{n\to\infty} \frac{A_{n,2k}}{n^k}, \quad k \geq 0,$$

we have $B_0 = 1$, and using (17) we find

$$B_k = \sum_{i=1}^{k} \gamma_i B_{k-i}, \quad k \geq 1$$

i.e.,

$$B_k = \sum_{i=1}^{k} \tfrac{(-1)^{i+1}\lambda^i}{i!}\, B_{k-i}, \quad k \geq 1. \tag{18}$$

Using (18) we can prove by mathematical induction that

$$B_k = \frac{\lambda^k}{k!}, \quad k \geq 0 \tag{19}$$

which completes the proof.

4. PROOF OF THEOREM 2

For arbitrary $x \in [-a, a]$ consider the function $g_x : [-a-1, a+1] \to \mathbb{R}$,

$$g_x := f - \sum_{i=0}^{2k} \tfrac{(e_1 - x e_0)^i}{i!}\, f^{(i)}(x).$$

Taylor's formula implies the existence of a point $\xi \in (-a-1, a+1)$, $|x-\xi| \leq |x-t|$, such that

$$g_x(t) = \frac{(t-x)^{2k}}{(2k)!}\,(f^{(2k)}(\xi) - f^{(2k)}(x)).$$

For any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|g_x(t)| \leq (t-x)^{2k}\,\varepsilon$$

for all $t \in [-a-1, a+1]$, $|t-x| < \delta$.

Let C be a constant such that $|g_x(t)| \leq C\,\delta^{2k+2}$, for all $x \in [-a, a]$, $t \in [-a-1, a+1]$. Consequently, we obtain

$$|g_x(t)| \leq \varepsilon\,(t-x)^{2k} + C\,(t-x)^{2k+2}$$

for all $x \in [-a, a]$, $t \in [-a-1, a+1]$, that is,

$$|g_x| \le \varepsilon\,(e_1 - xe_0)^{2k} + C\,(e_1 - xe_0)^{2k+2},$$

and so,

$$|L_n g_x(x)| \le \varepsilon\, L_n(e_1 - xe_0)^{2k}(x) + C\, L_n(e_1 - xe_0)^{2k+2}(x).$$

Using the equality

$$L_n(f)(x) = L_n(f \circ (e_1 + xe_0))(0),$$

we obtain

$$|L_n g_x(x)| \le \varepsilon\, L_n e_{2k}(0) + C\, L_n e_{2k+2}(0).$$

Taking into account the fact that

$$L_n e_i(0) = \frac{n!i!}{(n+i)!}\, A_{n,i}, \quad i = 1, 2, \ldots \tag{20}$$

it follows

$$\lim_{n\to\infty} n^i L_n e_{2i}(0) = \frac{\lambda^i}{i!}\,(2i)! \quad i = 1, 2, \ldots. \tag{21}$$

Consequently, we obtain

$$\lim_{n\to\infty} n^k L_n g_x(x) = 0,$$

uniformly for $x \in [-a, a]$, that is

$$\lim_{n\to\infty} n^k \left(L_n f(x) - \sum_{i=0}^{2k} \tfrac{L_n e_i(0)}{i!}\, f^{(i)}(x) \right) = 0.$$

Finally, using (16) the relation (8) is proved.

5. REMARKS

(a) Suppose that (1), (2) and (5) are satisfied and let $f \in C[-a-1, a+1]$ be $2k$–times differentiable at $x \in [-a, a]$. By using the Lemma and [11, Corollary 2] we obtain

$$\lim_{n\to\infty} n^k \left(L_n f(x) - \sum_{i=0}^{k-1} \tfrac{L_n e_{2i}(0)}{(2i)!}\, f^{(2i)}(x) \right) = \frac{\lambda^k}{k!} f^{(2k)}(x). \tag{22}$$

(b) If $x_{n,i} = -1 + 2i/n$, $i = 0, \ldots, n$, then $\lambda = 1/6$; in this special case the formula (22) can be found in [2]. In particular, for $k = 1$ and $k = 3$, we have

$$\lim_{n\to\infty} n(L_n f(x) - f(x)) = \tfrac{1}{6} f''(x), \tag{23}$$

respectively

$$\lim_{n\to\infty} n\left(n\left(n\big(L_n f(x)-f(x)\big)-\frac{f''(x)}{6}\right)-\frac{f^{IV}(x)}{72}\right)=\frac{f^{VI}(x)}{1296}-\frac{f^{IV}(x)}{180}. \tag{24}$$

(c) In the case of Chebyshev's knots

$$x_{n,k}=\cos^2\tfrac{2k+1}{2(n+1)}\,\pi,\quad k=0,\dots,n,$$

we obtain

$$\tfrac{1}{n+1}\sum_{k=0}^{n}\cos^2\tfrac{2k+1}{2(n+1)}\,\pi=\tfrac{1}{2},\quad \forall\, n\geq 1,$$

hence $\lambda=1$.

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