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INDUCED CONVEXITY AND THE PROBLEM OF THE INDUCED BEST APPROXIMATION

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In this paper the problem of the "induced best approximation of an element" of an arbitrary set X by elements of an induced seg-convex subset A of X is discussed.

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Everyday life freqently sets in front of us situations in which an object from a set A is replaced by another object a' from a set B, a' being obtained from a as its image by means of a transformation $T : A \to B$, therefore $a' = T(a)$. The human perception provides us with the most known example: every object is transformed, through the senses, into a set of impulses. The synthesis of the impulses generates an "image" of that object at the level of the brain. An important problem arises in these situations: for every property p of the object a, a property p' of the object a' is sought for, such that, for every object b from A having the property p, the object $T(b)$ has the property p'. In this case, the property p' is said to be the image by T of the property p. The problem of the description (identification) of the property p' is called recognition problem.

But there are situations in which the elements of the image have a property that is not directly noticeable in the case of the elements of the set A. In this type of cases the converse procedure is followed, identifying the property p'' of the elements of A satisfying the condition that for every object b of the image set B, which has the property p', the element $T^{-1}(b)$ has the property p''.

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In this situation the property p'' will be called induced property by the transformation T, and the problem of the description of the property p'' will be called the converse recognition problem. Both the recognition problem and its converse have been formulated in [2].

In this context, in the present paper we introduce the notion of "the induced best approximation point" (i.b.a.p.) of an element x^0 from a set X by elements from a nonempty subset A of X. For this purpose, a function $f : X \to Y$, $(Y, +, \cdot, \| \ \|)$ being an H-normed space, is used. After that, the properties of the induced best approximation points of x^0 by elements of A are studied in the case in which A is an induced seg-convex set (see [1]). This type of approximation accompanies both the recognition and the converse recognition problems, and in totally different and more general context was also formulated and partially solved in [2].

Let X be an arbitrary set, $(Y, +, \cdot, \| \ \|)$ be an H-normed linear space and a function $f : X \to Y$. Let be $x^0 \in X$ and $A \subseteq X$, $A \neq \emptyset$.

Definition 1. *A point $a^0 \in A$ is said to be $(f, Y)-$ induced element of the best approximation of x^0 by elements of A if*

$$\| f(a^0) - f(x^0) \| \leq \| f(a) - f(x^0) \| \quad \textit{for all} \quad a \in A. \tag{1}$$

We recall that in the classical case, i.e. $X = \mathbb{R}^n$, if $A \subseteq \mathbb{R}^n$ is a convex set and if y^0 is a given point of $\mathbb{R}^n$, then there exists at most one element of the best approximation (in classical sense) of y^0 by elements of A. In what follows, we show that this property of convex sets remains true under additional hypothesis, if the set A is not convex, but it is induced seg-convex (see [1]).

Let be $a \in A$ and $b \in A$. Supposing that the function f is injective, we can define the set

$$[a, b]_f \ = \ f^{-1}(\{tf(a) + (1 - t)f(b) \,|\, t \in [0, 1]\}. \tag{2}$$

In what follows, we will assume that the function f is injective.

Definition 2. *(see [1, Definition 6]). A subset $A \subseteq X$ is said to be induced seg-convex set with respect to f if*

$$[a, b]_f \ \subseteq \ A \quad \textit{for all}\, a \in A \,\textit{and}\, b \in A. \tag{3}$$

Theorem 1. *If the function $f : X \to Y$ is injective, x^0 is a given point of X, and A is a nonempty induced seg-convex set with respect to f such that $f(A)$ is a convex set in Y, then there exists at most one element of the (f, Y)-induced best approximation of x^0 by elements of A.*

Proof. Two cases are possible:

(1) $x^0 \in A$. Then, from the injectivity of f, it follows that x^0 is the single (f, Y) - induced best approximation point of x^0 by elements of A.

(2) $x^0 \notin A$. We suppose that there exist at least two elements a^0 and a of the

(f, Y) - induced best approximation of x^0 by elements of A. Then we have

$$\| f(a) - f(x^0) \| = \| f(a^0) - f(x^0) \| \neq 0. \tag{4}$$

Because f is an injective function, $f(a) \neq f(a^0)$. Then $\{f(a), f(a^0)\} \subset [f(a), f(a^0)] \subseteq f(A)$. It follows that there exists $\gamma \in f(A) \setminus \{f(a), f(a^0)\})$, such that $\gamma = \frac{1}{2} f(a) + \frac{1}{2} f(a^0)$. Since the set A is induced seg-convex with respect to f, we get $f^{-1}(\gamma) \in A$. Then there exists $c \in A$ such that

$$\gamma = f(c) = \tfrac{1}{2} f(a^0) + \tfrac{1}{2} f(a). \tag{5}$$

From (4) and (5) we have

$$\begin{aligned} \| f(c) - f(x^0) \|^2 &= \tfrac{1}{4} \| (f(a^0) - f(x^0)) + (f(a) - f(x^0)) \|^2 \\ &< \tfrac{1}{4} (\| (f(a^0) - f(x^0)) + (f(a) - f(x^0)) \|^2 + \\ &\quad + \| (f(a^0) - f(x^0)) - (f(a) - f(x^0)) \|^2) \end{aligned}$$

Applying the parallelograms equality for the H-norm we get

$$\begin{aligned} &\|(f(a^0) - f(x^0)) + (f(a) - f(x^0))\|^2 + \|(f(a^0) - f(x^0)) - (f(a) - f(x^0))\|^2 = \\ &\qquad = 2\|f(a^0) - f(x^0)\|^2 + 2\|f(a) - f(x^0)\|^2. \end{aligned}$$

Then, in view of (4), we obtain $\|(f(c) - f(x^0)\| < \|f(a^0) - f(x^0)\|$.
This contradicts the fact that a^0 is an element of the (f, Y) - induced best approximation of x^0 by elements of A. □

Remark 1. If $f(A)$ is a non-convex set in Y, then the conclusion of the theorem 1 is not always true.

Example 1. Let be $X = \{(0, 10), (0, 0), (1, 0)\}$, $Y = \mathbb{R}^2$, $x^0 = (0, 0)$, $A = \{(0, 1), (1, 0)\}$. If we take $f : X \to Y$, $f(x_1, x_2) = (x_1, x_2)$, for each $(x_1, x_2) \in X$, then it is easy to see that the set A is induced seg-convex with respect to f. But we have that both $(0, 1)$ and $(1, 0)$ are, in the same time, elements of the (f, Y)-induced best approximation of x^0 by elements of A.

LEMMA 1. *If the function $f : X \to Y$ is injective, if the set $f(X)$ is convex (in classical sense) and if the set A is induced seg-convex with respect to f, then the set $f(A)$ is also convex (in classical sense).*

Proof. Let be u', $u'' \in f(A)$. There exist $a\prime$, $a'' \in A$ such that $u' = f(a')$, and $u'' = f(a'')$. Let $t \in]0,1[$. Because the set $f(X)$ is convex, we have $u = tu' + (1-t)u'' \in f(X)$. From the induced convexity of A, it follows that $f^{-1}(u) \in f^{-1}([f(a'), f(a'')] \subseteq A$. Then there exists $a \in A$, such that $f(a) = u$. We get that $u = tf(a') + (1-t)f(a'') \in f(A)$. Because t is arbitrarily chosen in $]0,1[$, it results that A is a convex set in Y. □

From theorem 1 and lemma 2 we get:

COROLLARY 1. *If the function $f : X \to Y$ is injective, $f(X)$ is a convex set, x^0 is a given point of X, and A is a nonempty induced seg-convex set with respect to f, then there exists at most one element of the (f, Y)-induced best approximation of x^0 by elements of A.*

THEOREM 2. *If the function $f : X \to Y$ is injective, x^0 is a given point of X, and if A is a nonempty induced seg-convex set with respect to f, then the set $A(x^0)$ of all the elements of (f, Y)-induced best approximation of x^0 by elements of A is also induced seg-convex with respect to f.*

Proof. Two cases are possible:

(1) $card(A(x^0)) \in \{0,1\}$; obviously $A(x^0)$ is an induced seg-convex set.
(2) $card(A(x^0)) > 1$. Let be a', $a'' \in A(x^0)$, and

$$\lambda = \| f(x^0) - f(a') \| = \| f(x^0) - f(a'') \| .$$

Let be $a \in f^{-1}([f(a'), f(a'')])$. Then there exists $t \in [0,1]$ such that $f(a) = tf(a') + (1-t)f(a'')$. We have

$$\begin{aligned} \|f(x^0) - f(a)\| &= \|t(f(x^0) - f(a')) + (1-t)(f(x^0) - f(a''))\| \\ &\leq t\|f(x^0) - f(a')\| + (1-t)\|f(x^0) - f(a'')\| \\ &= \lambda. \end{aligned}$$

On the other hand, we have $\|f(x^0) - f(a)\| \geq \lambda$. It follows that $\|f(x^0) - f(a)\| = \lambda$. Then $a \in A(x^0)$. Hence the set $A(x^0)$ is induced seg-convex with respect to f. □

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