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EXTENSIONS OF SEMI-LIPSCHITZ FUNCTIONS ON QUASI-METRIC SPACES

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. The aim of this note is to prove an extension theorem for semi-Lipschitz real functions defined on quasi-metric spaces, similar to McShane extension theorem for real-valued Lipschitz functions defined on a metric space ([2], [4]).

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1. INTRODUCTION

Let X be a nonvoid set. A *quasi-metric* on X is a function $d : X \times X \to [0, \infty)$ satisfying the conditions

(i) $$d(x, y) = d(y, x) = 0 \Longleftrightarrow x = y; \quad x, y \in X,$$

(ii) $$d(x, y) \leq d(x, z) + d(z, y), \quad x, y, z \in X.$$

If d is a quasi-metric on X, then the pair (X, d) is called a *quasi-metric space*.

The conjugate of quasi-metric d, denoted by d^{-1} is defined by $d^{-1}(x, y) = d(y, x)$, $x, y \in X$.

Obviously the function $d^s : X \times X \to [0, \infty)$ defined by

$$d^s(x, y) = \max\left\{d(x, y), d^{-1}(x, y)\right\}; \quad x, y \in X$$

is a metric on X.

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If the quasi-metric d can take the value $+\infty$, then it is called an *extended quasi-metric*.

Let (X,d) be a quasi-metric space. A function $f:X\to\mathbb{R}$ is called *semi-Lipschitz* if there exists a constant $K\geq 0$ so that

$$f(x)-f(y)\leq K\cdot d(x,y)\,, \tag{1}$$

for all $x,y\in X$. The number $K\geq 0$ in (1) is called a semi-Lipschitz constant for f.

For a quasi-metric space (X,d) the real-valued function $f:X\to\mathbb{R}$ is said to be $\leq_d$*-increasing if*

$$d(x,y)=0 \quad \text{implies} \quad f(x)-f(y)\leq 0,\quad x,y\in X \tag{2}$$

or equivalently,

$$f(x)-f(y)>0 \quad \text{implies} \quad d(x,y)>0,\quad x,y\in X. \tag{3}$$

Note that every semi-Lipschitz function on quasi-metric space (X,d) is $\leq_d$-*increasing* (see (1)).

For a semi-Lipschitz function $f:X\to\mathbb{R}$, where (X,d) is a quasi-metric space, denote by $\|f\|_d$ the constant:

$$\|f\|_d=\sup\left\{\frac{(f(x)-f(y))\vee 0}{d(x,y)}:d(x,y)>0,\quad x,y\in X\right\}. \tag{4}$$

Theorem 1. *Let (X,d) a quasi-metric space and $f:X\to\mathbb{R}$ a semi-Lipschitz function. Then $\|f\|_d$ defined by (4) is the smallest semi-Lipschitz constant for f.*

Proof. If $f:X\to\mathbb{R}$ is semi-Lipschitz, then f is $\leq_d$-increasing, and then $f(x)-f(y)>0$ implies $d(x,y)>0$. It follows that

$$\frac{(f(x)-f(y))\vee 0}{d(x,y)}=\frac{f(x)-f(y)}{d(x,y)}>0.$$

The inequalities $f(x)-f(y)\leq 0$ and $d(x,y)>0$ imply

$$\frac{(f(x)-f(y))\vee 0}{d(x,y)}=0.$$

Consequently $\|f\|_d\geq 0$.

For $f(x)-f(y)<0$ it follows $(f(x)-f(y))/d(x,y)\leq\|f\|_d$ and obviously for $f(x)-f(y)\leq 0$ we have $f(x)-f(y)\leq 0\leq\|f\|_d\cdot d(x,y)$.

Consequently

$$f(x) - f(y) \leq \|f\|_d \cdot d(x, y)$$

for all $x, y \in X$.

Now let $K \geq 0$ such that

$$f(x) - f(y) \leq K \cdot d(x, y), \quad \text{for all } x, y \in X.$$

The function f is $\leq_d$-increasing, and then

$$\frac{(f(x) - f(y)) \vee 0}{d(x, y)} = \begin{cases} \frac{f(x)-f(y)}{d(x,y)} \leq K, & \text{if } f(x) - f(y) > 0, \\ 0 \leq K, & \text{if } f(x) - f(y) \leq 0, \end{cases}$$

Consequently $\|f\|_d \leq K$. □

For a quasi-metric (X, d) let us consider the set:

(5)
$$S\,Lip\,X = \left\{ f : X \to \mathbb{R} \mid f \text{ is } \leq_d\text{-increasing}, \sup_{d(x,y)\neq 0} \frac{(f(x)-f(y))\vee 0}{d(x,y)} < \infty \right\}.$$

It is straightforward to see that $S\,Lip\,X$ is exactly the set of all semi-Lipschitz functions on (X, d) (see [6]).

2. EXTENSIONS OF SEMI-LIPSCHITZ FUNCTIONS

Let $Y \subset X$ where (X, d) is a quasi-metric space. Then (Y, d) is a quasi-metric space with the quasi-metric induced by d (denoted by d too). Let us denote by $S\,Lip\,Y$ the set of all semi-Lipschitz functions defined on Y and let

(6)
$$\|f\|_d = \sup \left\{ \frac{(f(x) - f(y)) \vee 0}{d(x, y)} : x, y \in Y,\ d(x, y) \neq 0 \right\}$$

be the smallest semi-Lipschitz constant for $f \in S\,Lip\,Y$.

If $f \in S\,Lip\,Y$, a function $F \in S\,Lip\,X$ is called an *extension* (preserving the smallest semi-Lipschitz constant) of f if:

(7)
$$F|_Y = f \quad \text{and} \quad \|F\|_d = \|f\|_d\,.$$

Denote by $E_Y(f)$ the set of all extensions of the function $f \in S\,Lip\,Y$, i.e.

(8)
$$E_Y(f) = \left\{ F \in S\,Lip\,X : F|_Y = f \text{ and } \|F\|_d = \|f\|_d \right\}$$

THEOREM 2. *Let (X, d) be a quasi-metric space and Y a nonvoid subset of X. Then for every $f \in S\,Lip\,Y$ the set $E_Y(f)$ is nonvoid.*

Proof. Let $f \in S\,Lip\,Y$ and the constant $\|f\|_d$ defined by (6).
Consider the function

$$F(x) = \inf_{y\in Y} \{f(y) + \|f\|_d\, d(x,y)\}, \; x \in X. \tag{9}$$

a) *First we show that F is well defined.*
Let $z \in Y$and $x \in X$.For any $y \in Y$ we have

$$\begin{aligned} f(y) + \|f\|_d\, d(x,y) &= f(z) + \|f\|_d\, d(x,y) - (f(z) - f(y)) \\ &\geq f(z) + \|f\|_d\, d(x,y) - \|f\|_d\, d(z,y) \\ &= f(z) - \|f\|_d\, (d(z,y) - d(x,y)). \end{aligned}$$

The inequality $d(z,y) - d(x,y) \leq d(z,x) = d^{-1}(x,z)$ implies

$$f(y) + \|f\|_d\, d(x,y) \geq f(z) - \|f\|_d \cdot d^{-1}(x,z) \tag{10}$$

showing that for every $x \in X$ the set $\{f(y) + \|f\|_d\, d(x,y) : y \in Y\}$ is bounded from above by $f(z) - \|f\|_d\, d^{-1}(x,z)$, and the infimum (9) is finite.
b) *We show now that $F(y) = f(y)$ for all $y \in Y$.*
Let $y \in Y$. Then

$$F(y) \leq f(y) + \|f\|_d\, d(y,y) = f(y).$$

For any $v \in Y$ we have

$$f(y) - f(v) \leq \|f\|_d \cdot d(y,v)$$

so that

$$f(v) + \|f\|_d \cdot d(y,v) \geq f(y)$$

and

$$F(y) = \inf\{f(v) + \|f\|_d\, d(y,v) : v \in Y\} \geq f(y).$$

It follows $F(y) = f(y)$.

c) *We prove that $\|F\|_d = \|f\|_d$.*
Since $F|_Y = f$, the definitions of $\|F\|_d$ and $\|f\|_d$ yield $\|F\|_d \geq \|f\|_d$.
Let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$F(x_1) \geq f(y) + \|f\|_d\, d(x_1,y) - \varepsilon$$

we obtain

$$\begin{aligned} F(x_2) - F(x_1) &\leq f(y) + \|f\|_d\, d(x_2,y) - (f(y) + \|f\|_d \cdot d(x_1,y) - \varepsilon) \\ &= \|f\|_d\, [d(x_2,y) - d(x_1,y)] + \varepsilon \\ &\leq \|f\|_d \cdot d(x_2,x_1) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$F(x_2) - F(x_1) \le \|f\|_d \cdot d(x_2, x_1)$$

for any $x_1, x_2 \in X$ and $\|F\|_d \le \|f\|_d$.

d) *The function F is $\le_d$-increasing.*

Indeed, let be $u, v \in X$ and $d(u,v) = 0$. We have $d(u,y) \le d(u,v) + d(v,y)$. Consequently

$$d(u,y) \le d(v,y).$$

Then

$$f(y) + \|f\|_d \, d(u,y) \le f(y) + \|f\|_d \, d(v,y).$$

It follows that

$$F(u) \le F(v),$$

and consequently $d(u,v) = 0$ implies $F(u) \le F(v)$.

It follows that $F \in E_Y^d(f)$ so that $E_Y^d(t) \ne \emptyset$. □

REMARKS 1. 1^0 Similarly, the function

$$G(x) = \sup_{y \in Y} \left\{ f(y) - \|f\|_d \, d^{-1}(x,y) \right\} \tag{11}$$

is $\le_d$-increasing, and G belongs to $E_Y^d(f)$ too.

2^0 The inequality

$$G(x) \le F(x), \tag{12}$$

holds for every $x \in X$.

Indeed, taking the infimum with respect to $z \in Y$ and then the supremum with respect to $y \in Y$ in (10) we find

$$G(x) = \sup_{y \in Y} \left\{ f(y) - \|f\|_d \, d^{-1}(x,y) \right\} \le \inf_{z \in Y} \{ f(z) + \|f\|_d \, d(x,z) \} = F(x).$$

In fact, the following theorem holds:

THEOREM 3. *Let (X,d) be a quasi-metric space, Y a nonvoid subset of X and $f \in S\,Lip\,Y$.*

Then for any $H \in E_Y^d(f)$ we have

$$G(x) \le H(x) \le F(x), \quad x \in X. \tag{13}$$

Proof. Let $H \in E_Y^d(f)$. For arbitrary $x \in X$ and $y \in Y$ we have

$$H(x) - H(y) \le \|f\|_d \, d(x,y)$$

implying

$$H(x) \le H(y) + \|f\|_d \, d(x,y) = f(y) + \|f\|_d \,(x,y).$$

Taking the imfimum with respect to $y \in Y$ we get

$$H(x) \le \inf_{y \in Y} \left\{ f(y) + \|f\|_d \, d(x,y) \right\} = F(x).$$

The inequality $H(x) \ge G(x)$, $x \in X$ can be proved similarly. □

Corollary 4. *A function $f \in S\,Lip\,Y$ has a unique extension in $S\,Lip\,X$ if and only if the following relation*

$$\inf_{y \in Y} \{ f(y) + \|f\|_d \, d(x,y) \} = \sup_{y \in Y} \{ f(y) - \|f\| \, d(y,x) \}, \tag{14}$$

holds for every $x \in X$.

Example.

Let $\mathbb{R}$ be the real axis and $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ the quasi-metric defined by

$$d(x,y) = \begin{cases} x - y, & \text{if } \; x \ge y \\ 1, & \text{if } \; x < y. \end{cases}$$

Let Y be given by $Y = [0,1] \subset \mathbb{R}$ and $f : Y \to \mathbb{R}$, $f(y) = 2y$. Then f is semi-Lipschitz on Y and $\|f\|_d = 2$. The extension F defined by (9) is

$$F(x) = \begin{cases} 2, & \text{if } \; x < 0 \\ 2x, & \text{if } \; x \ge 0 \end{cases}$$

and the extension G defined by (11) is

$$G(x) = \begin{cases} 2x, & x \le 1 \\ 0, & x > 1 \end{cases}$$

Obviously, $G(x) \le F(x)$, $x \in \mathbb{R}$.

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