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ON A HALLEY-STEFFENSEN METHOD FOR APPROXIMATING THE SOLUTIONS OF SCALAR EQUATIONS

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Dedicated to the memory of Acad. Tiberiu Popoviciu

Abstract. In the present paper we show that the Steffensen method for solving the scalar equation $f(x) = 0$, applied to equation

$$h(x) = \frac{f(x)}{\sqrt{f'(x)}} = 0,$$

leads to bilateral approximations for the solution. Moreover, the convergence order is at least 3, i.e. as in the case of the Halley method.

MSC 2000. 65H05.

1. INTRODUCTION

Let $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$ be given. Assume that $f \in C^4[a, b]$ and $f'(x) > 0$, $\forall x \in [a, b]$. Consider the function $h : [a, b] \to \mathbb{R}$ given by $h(x) = f(x) / \sqrt{f'(x)}$. As it is well known (see e.g. [2]), the Halley method for solving

$$f(x) = 0 \tag{1.1}$$

consists in constructing the sequence $(x_n)_{n\geq 0}$ by

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}, \qquad n = 0, 1, \ldots, \; x_0 \in [a, b] \tag{1.2}$$

As it can be easily noticed, (1.2) is the Newton method applied to equation $h(x) = 0$. The advantage of using the function h instead of f in (1.2) consists

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in the fact that h obeys $h''(\bar{x}) = 0$, where $\bar{x}$ is a solution of (1.1). It is well known that $h''(\bar{x}) = 0$ ensures for (1.2) the convergence order 3 (see [2]).

Starting from an algorithm proposed by Heron for approximating $\sqrt[3]{100}$, the authors of [4] establish the following relation:

$$\sqrt[3]{N} \simeq \Phi(N;\alpha,\beta) = \alpha + \tfrac{\beta d_1}{\beta d_1 + \alpha d_2}(\beta - \alpha) \tag{1.3}$$

where $d_1 = N - \alpha^3$, $d_2 = \beta^3 - N$ and $\alpha^3 \leq N \leq \beta^3$. The Heron algorithm is obtained from (1.3) for $\alpha = 4$ and $\beta = 5$.

In the paper [5], the authors noticed that the approximation given by (1.3) for $\sqrt[3]{N}$ is obtained by applying one step of the chord method to equation $\varphi(x) = 0$, where $\varphi(x) = x^2 - N/x$, $x > 0$.

In [5] it is noticed that if one considers equation $f(x) = x^3 - N$, then, apart of a constant factor, equation $\varphi(x) = 0$ is equivalent to

$$h(x) = \frac{f(x)}{\sqrt{f'(x)}} = 0, \qquad x > 0,$$

i.e., $1/\sqrt{3}\left(x^2 - N/x\right) = 0$. Therefore there exists a connection between the Halley method (1.2) and the Heron algorithm concerning the function h. In the case of the Halley method there is applied the Newton method to $h(x) = 0$ whereas in the Heron method there is applied the chord method to $h(x) = 0$. The both methods benefit from the advantages implied by $h''(\bar{x}) = 0$. These remarks have led in [5] to generalizations of the results from [4]. A brief analysis of the convergence order of the chord method applied to $h(x) = 0$ in the general case (quasi-Halley method) is given in [1].

In this note we shall study the convergence of an iterative method obtained by applying the Steffensen algorithm to equation $h(x) = 0$, where $h(x) = f(x)/\sqrt{f'(x)}$. As we shall see, this method has some advantages over the Halley and chord methods applied to h. The most important one is the fact that the method we propose allows the control of the absolute error at each iteration step. Its convergence order is the same as for the Halley method, being higher than the order of the chord method.

For solving (1.1) we shall consider the sequence

$$x_{n+1} = x_n - \frac{h(x_n)}{[x_n, \varphi(x_n); h]}, \qquad n = 0, 1, \ldots,\ x_0 \in [a, b] \tag{1.4}$$

where φ will be suitably chosen, and $[x, y; f] = (f(y) - f(x))/(y - x)$ denotes the first order divided difference of f on x and y. We shall call this method the Halley-Steffensen method.

2. LOCAL CONVERGENCE AND ERROR BOUNDS

Concerning the function f we shall assume the following conditions:

i. $f \in C^4[a,b]$;
ii. $f'(x) > 0$ for all $x \in [a,b]$;
iii. equation (1.1) has a solution $\bar{x} \in]a,b[$;
iv. the function φ from (1.4) is given by

$$\varphi(x) = x - \frac{f(x)}{\lambda} \tag{2.1}$$

where $0 < \lambda < f'_d(a)$, and $f'_d(a)$ is the right derivative of f at a;
v. $f'(x) < 2\lambda,\ \forall x \in [a,b]$;
vi. $f''(x) > 0,\ \forall x \in [a,b]$.

We notice in the beginning that

$$x_n - \frac{h(x_n)}{[x_n, \varphi(x_n); h]} = \varphi(x_n) - \frac{h(\varphi(x_n))}{[x_n, \varphi(x_n); h]}, \qquad n = 0, 1, \ldots$$

which shows that the Halley-Steffensen sequence obeys

$$x_{n+1} = \varphi(x_n) - \frac{h(\varphi(x_n))}{[x_n, \varphi(x_n); h]}, \qquad n = 0, 1, \ldots, \tag{2.2}$$

while for the first and second order derivatives we obtain

$$h'(x) = \tfrac{2(f'(x))^2 - f''(x)f(x)}{2(f'(x))^{3/2}}; \tag{2.3}$$

$$h''(x) = \tfrac{3(f''(x))^2 - 2f'''(x)f'(x)}{4(f'(x))^{5/2}} f(x). \tag{2.4}$$

Relation (2.4) implies $h''(\bar{x}) = 0$, while (2.3) implies $h'(\bar{x}) = (f'(\bar{x}))^{1/2} > 0$. Since h' is continuous and $h'(\bar{x}) > 0$ it follows the existence of $\alpha, \beta \in \mathbb{R}$, $a \le \alpha < \bar{x} < \beta \le b$, such that $h'(x) > 0,\ \forall x \in [\alpha, \beta]$.

We obtain the following result:

THEOREM 1. *Assume that the function f and the initial approximation x_0 satisfy:*

i$_1$. *the number* x_0 *is sufficiently close to* $\bar{x}$ *and* $\varphi(x_0) \in [\alpha, \beta]$, *with* α *and* β *determined above;*
ii$_2$. *the function* f *obeys* (i)–(vi);

Then the Halley-Steffensen sequence (1.4) *converges to the solution* $\bar{x}$ *and, moreover,*

$$
\begin{aligned}
&(2.5) \qquad |x_{n+1} - \bar{x}| \leq \max\left\{ |x_{n+1} - x_n|, |x_{n+1} - \varphi(x_n)| \right\}, \qquad n = 0, 1, \ldots, \\
&(2.6) \qquad |x_{n+1} - \bar{x}| \leq K |x_n - \bar{x}|^3, \qquad n = 0, 1, \ldots,
\end{aligned}
$$

where K *is a constant which does not depend on* n.

Proof. Since $f''(x) > 0$, $\forall x \in [a, b]$, it follows that f' is increasing on $[a, b]$ and hence $\varphi'(x) = 1 - f'(x)/\lambda$ obeys $\varphi'(x) < 0$, $\forall x \in [a, b]$. The existence of the interval $[\alpha, \beta] \subseteq [a, b]$ such that $h'(x) > 0$, $\forall x \in [\alpha, \beta]$, has been proved above. Assumption iv) implies $\bar{x} = \varphi(\bar{x})$ and so, if $x_0 < \bar{x}$, we get $\varphi(x_0) > \bar{x}$. Analogously, $x_0 > \bar{x} \Rightarrow \varphi(x_0) < \bar{x}$, i.e., $\bar{x} \in I_0$, I_0 being the interval determined by x_0 and $\varphi(x_0)$. From $h'(x) > 0$, $\forall x \in [\alpha, \beta]$, it follows $h(x_0) < 0$ for $x_0 < \bar{x}$ resp. $h(x_0) > 0$ for $x_0 > \bar{x}$. Relation (1.4) implies $x_1 > x_0$ when $x_0 < \bar{x}$ resp. $x_1 < x_0$ when $x_0 > \bar{x}$. By (2.2) we get $x_1 < \varphi(x_0)$ when $x_0 < \bar{x}$ resp. $x_1 > \varphi(x_0)$ when $x_0 > \bar{x}$, i.e. $x_1 \in I_0$. We shall show that $\varphi(x_1) \in I_0$. For this we prove that $x_0 < \varphi(\varphi(x_0))$ when $x_0 < \bar{x}$ resp. $x_0 > \varphi(\varphi(x_0))$ when $x_0 > \bar{x}$. For $\varphi(\varphi(x_0))$ we easily obtain the following expression:

$$
\varphi(\varphi(x_0)) = x_0 - \tfrac{2}{\lambda} f(x_0) + \tfrac{1}{\lambda^2} f'(\xi_0) f(x_0), \ \xi_0 \in I_0,
$$

whence, taking into account (v) and $f(x_0) < 0$ for $x_0 < \bar{x}$, we obtain $\varphi(\varphi(x_0)) - x_0 > 0$. Analogously, if $x_0 > \bar{x}$ then $\varphi(\varphi(x_0)) - x_0 < 0$. So, $\varphi(\varphi(x_0)) \in I_0$. Let $x_0 < \bar{x}$, and so $\varphi(x_0) > \bar{x}$ and $x_0 < x_1 < \varphi(x_0)$. Since φ is decreasing we get $\varphi(x_1) > \varphi(\varphi(x_0)) > x_0$, and, on the other hand, from $x_0 < x_1 \Rightarrow \varphi(x_0) > \varphi(x_1)$, i.e. $\varphi(x_1) \in I_0$. Analogously, if $x_0 > \bar{x} \Rightarrow \varphi(x_1) \in I_0$. Denoting by I_1 the closed interval determined by x_1 and $\varphi(x_1)$ then

$$
\bar{x} \in I_1 \text{ and } I_0 \supset I_1. \tag{2.7}
$$

Let I_s denote the closed intervals determined by the points x_s and $\varphi(x_s)$, $s = \overline{0, k}$. Suppose that

$$I_0 \supset I_1 \supset \ldots \supset I_{k-1} \supset I_k, \tag{2.8}$$

and $\bar{x} \in I_k$. As we have shown for x_0, we can prove that the interval I_{k+1}, determined by x_{k+1} and $\varphi(x_{k+1})$, obeys

$$I_k \supset I_{k+1}$$

and $\bar{x} \in I_{k+1}$. From the above reasons, it follows that relations (2.5) are true. It remains to show that (2.6) holds, which implies the convergence of $(x_n)_{n\geq 0}$ generated by (1.4). For this purpose we shall use the identity

$$h(\bar{x}) = h(x_n) + [x_n, \varphi(x_n); h](\bar{x} - x_n) + [\bar{x}, x_n, \varphi(x_n); h](\bar{x} - x_n)(\bar{x} - \varphi(x_n)),$$

whence, taking into account (1.4) and $h(\bar{x}) = 0$, it follows (see [9])

$$\bar{x} - x_{n+1} = -\frac{[\bar{x}, x_n, \varphi(x_n); h]}{[x_n, \varphi(x_n); h]}(\bar{x} - x_n)(\bar{x} - \varphi(x_n)), \quad n = 0, 1, \ldots \tag{2.9}$$

By (ii) and (v) we get $|\varphi'(x)| < 1$, $\forall x \in I_0$, and so

$$|\bar{x} - \varphi(x_n)| < |\bar{x} - x_n|, \qquad n = 0, 1, \ldots \tag{2.10}$$

On the other hand, from the mean value theorems for divided differences one obtains

$$[\bar{x}, x_n, \varphi(x_n); h] = \tfrac{h''(\xi_n)}{2!}, \quad \xi_n \in I_n, \quad \text{and} \tag{2.11}$$

$$[x_n, \varphi(x_n); h] = h'(\eta_n), \quad \eta_n \in I_n. \tag{2.12}$$

From (2.11) and from $h''(\bar{x}) = 0$ we get

$$\big|[\bar{x}, x_n, \varphi(x_n); h]\big| = \tfrac{1}{2}\left|h''(\xi_n) - h''(\bar{x})\right| = \tfrac{1}{2}h'''(\delta_n)\,|\xi_n - \bar{x}|, \tag{2.13}$$

where $\delta_n \in I_n$. Further, by (2.10) it follows

$$|\xi_n - \bar{x}| \leq \max\left\{ |\bar{x} - x_n|, |\bar{x} - \varphi(x_n)| \right\} \leq |\bar{x} - x_n|. \tag{2.14}$$

Denote $m_3 = \sup_{x\in I_0} |h'''(x)|$ and $m_1 = \inf_{x\in I_0} |h'(x)|$. Relations (2.9)–(2.14) lead to

$$|x_{n+1} - \bar{x}| \leq \tfrac{m_3}{2m_1}\,|x_n - \bar{x}|^3, \quad n = 0, 1, \ldots,$$

i.e., (2.6) with $K = m_3/(2m_1)$. The proof is complete. □

3. NUMERICAL EXAMPLE

Consider the equation

$$f(x) = x^3 - 20, \quad x > 0,$$

and the function $h(x) = \frac{1}{\sqrt{3}}(x^2 - 20/x)$. Obviously, $h'(x) = 2\left(x + 10/x^2\right) > 0$, $\forall x > 0$. It can be easily seen that $2.6 < \sqrt[3]{20} < 2.8$. We choose $\varphi(x) = x - (x^3 - 20)/20.28$. Then, for $x_0 = 2.6$, $\varphi(x_0) < 2.8$, and hence our theorem may be applied. We obtain the following results:

n	x_n	$\varphi(x_n)$	$h(x_n)$
0	$2{,}600\,000\,000\,0 \cdot 10^{+00}$	$2{,}719\,526\,627\,2 \cdot 10^{+00}$	$-9{,}323\,076\,923\,2 \cdot 10^{-01}$
1	$2{,}714\,420\,633\,0 \cdot 10^{+00}$	$2{,}714\,417\,345\,3 \cdot 10^{+00}$	$2{,}456\,359\,652\,6 \cdot 10^{-05}$
2	$2{,}714\,417\,616\,6 \cdot 10^{+00}$	$2{,}714\,417\,616\,6 \cdot 10^{+00}$	$-1{,}455\,191\,522\,8 \cdot 10^{-11}$

REFERENCES

[1] Adi Ben-Israel, *Newton's method with modified functions*, Contemp. Math., 204 (1997), pp. 39–50.

[2] G. H. Brown, Jr., *On Halley's variation of Newton's method*, Amer. Math. Monthly, 84 (1977), 726–728.

[3] V. Candela and A. Marquina, *Recurrence relations for rational cubic methods I: the Halley's method*, Computing 44 (1990), 169–184.

[4] G. Deslauries and S. Dubuc, *Le calcul de la racine cubique selon Héron*, El. Math., 51 (1996), 28–34.

clickable → [5] D. Luca and I. Păvăloiu, *On the Heron's method for the approximation of the cubic root of a real number*, Rev. Anal. Numér. Théor. Approx., 28 (1997), 103–108.

[6] A. M. Ostrowski, *The Solution of Equations and Systems of Equations*, Academic Press, New York–London, 1960.

clickable → [7] I. Păvăloiu, *On the monotonicity of the sequences of approximations obtained by Steffensen's method*, Mathematica (Cluj) 35 (58) (1993), 71–76.

clickable → [8] I. Păvăloiu, *Approximation of the roots of equations by Aitken-Steffensen-type monotonic sequences*, Calcolo, 32 (1995), 69–82.

available soon, refresh and click here → [9] T. Popoviciu, *Sur la délimitation de l'erreur dans l'approximation des racines d'une équation par interpolation linéaire ou quadratique*, Rev. Roumaine Math. Pures Appl., 13 (1968), 75–78.

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