REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 30 (2001) no. 2, pp. 219–227
ictp.acad.ro/jnaat

ON THE ARCLENGTH OF TRIGONOMETRIC INTERPOLANTS*

JÜRGEN PRESTIN[1] and EWALD QUAK[2]

Dedicated to Professor Dr. Werner Haussmann on his 60th birthday

Abstract. As pointed out recently by Strichartz [5], the arclength of the graph $\Gamma(S_N(f))$ of the partial sums $S_N(f)$ of the Fourier series of a jump function f grows with the order of $\log N$. In this paper we discuss the behaviour of the arclengths of the graphs of trigonometric interpolants to a jump function. Here the boundedness of the arclengths depends essentially on the fact whether the jump discontinuity is at an interpolation point or not. In addition convergence results for the arclengths of interpolants to smoother functions are presented.

MSC 2000. 41A15.

1. INTRODUCTION

The famous Gibbs phenomenon of overshooting is one of the well-known disadvantages of the Fourier series approach. It is closely related to the logarithmic growth of the $L^1_{2\pi}$-norm of the Dirichlet kernel, i.e. the Lebesgue constant for the Fourier partial sum operator. It can also be seen as one of the motivations for introducing different means of Fourier sums.

In the very recent paper [5], Strichartz investigated the behaviour of the arclengths of the graphs $\Gamma(S_N(f))$ of the partial sums $S_N(f)$ of the Fourier series of a piecewise smooth function f. It turns out that in the case of jump discontinuities, the arclengths of the graphs $\Gamma(S_N(f))$ tend to infinity with logarithmic order, while for continuous piecewise C^1 functions the arclength of $\Gamma(S_N(f))$ converges to the arclength of $\Gamma(f)$.

*Research of the authors supported by the EU Research Training Network MINGLE, RTN1-1999-00212.

[1]Medical University of Lübeck, Institute of Mathematics, D–23560 Lübeck, Germany, e-mail: `prestin@math.mu-luebeck.de`.

[2]SINTEF Applied Mathematics, P.O. Box 124 Blindern, N-0314 Oslo, Norway, e-mail: `ewald.quak@math.sintef.no`.

It is the aim of this paper to investigate analogous questions for trigonometric Lagrange interpolation.

Therefore we define for each positive integer N the trigonometric interpolant $L_N f$ to a given 2π-periodic function f by

$$L_N f(t) = \sum_{s=0}^{2N-1} f\left(\tfrac{s\pi}{N}\right) \varphi_N\left(t - \tfrac{s\pi}{N}\right),$$

where

$$\varphi_N(t) = \tfrac{1}{2N}\Big(1 + 2\sum_{k=1}^{N-1} \cos kt \; + \cos Nt\Big)$$

is a modified Dirichlet kernel.

Then $L_N f(\frac{k\pi}{N}) = f(\frac{k\pi}{N})$ holds for all integer k. As in the case of the Fourier sum we can restrict our attention to the 2π-periodic jump function

$$f_0(t) = \begin{cases} (\pi - t)/2, & \text{if} \quad 0 < t \le \pi, \\ 0, & \text{if} \quad t = 0, \\ (-\pi - t)/2, & \text{if} \quad -\pi \le t < 0. \end{cases}$$

With its Fourier expansion given as

$$f_0(t) = \sum_{\ell=1}^{\infty} \frac{\sin \ell t}{\ell},$$

this piecewise linear function is a standard example for the Gibbs phenomenon. The underlying idea is then to consider functions with finitely many jumps in the period interval as the sum of translates of f_0 and a smooth function.

Different from the case of Fourier sums, for the interpolation process it is important, however, to know whether the jump discontinuity is at an interpolation point or not. Therefore we distinguish between our jump test function f_0 and and its translates $f_\varepsilon(t) = f_0(t - \varepsilon)$, where $0 < \varepsilon < \frac{\pi}{N}$.

It turns out that the behaviour of the arclengths of the graphs $\Gamma(L_N(f_\varepsilon))$ depends essentially on the choice of ε. Namely, for $\varepsilon = 0$ we have bounded arclength and for $0 < \varepsilon < \frac{\pi}{N}$ the arclength behaves like $\log N$. Some overshoot, however, is always present also in the case of bounded arclengths, see Figures 1 and 2. Notice that the nice behaviour of the interpolant of f_0 not only stems from the fact that the jump discontinuity is at an interpolation point, but also from

$$f_0(0) = \frac{f_0(0-) + f_0(0+)}{2}. \tag{1}$$

If an arbitrary jump function does not satisfy (1), we have to add to the interpolation polynomial a multiple of φ_N, which results also in an unbounded arclength (cf. the proof of Theorem 3.1).

Finally we mention that the use of modified interpolation processes can improve the behaviour of the graphs of the interpolants essentially. In this note we restrict ourselves to certain de la Vallée Poussin kernels, which possess interesting features for generating corresponding wavelets (cf. [2], [3]).

2. THE INTERPOLANT OF THE JUMP FUNCTION f_ε

We start by stating some basic identities for discrete inner products of trigonometric functions.

LEMMA 2.1. *The following discrete orthogonality relations hold for all integers k, ℓ*

$$\begin{aligned}\sum_{s=0}^{2N-1} \sin \tfrac{\ell s\pi}{N} \cos \tfrac{ks\pi}{N} &= 0,\\ \sum_{s=0}^{2N-1} \cos \tfrac{\ell s\pi}{N} \cos \tfrac{ks\pi}{N} &= N \cdot (\tilde{\delta}_{\ell,k} + \tilde{\delta}_{\ell,-k})\end{aligned}$$

and

$$\sum_{s=0}^{2N-1} \sin \tfrac{\ell s\pi}{N} \sin \tfrac{ks\pi}{N} = N \cdot (\tilde{\delta}_{\ell,k} - \tilde{\delta}_{\ell,-k}),$$

where

$$\tilde{\delta}_{\ell,k} = \begin{cases} 1, & \ell \equiv k \mod 2N,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. These identities follow directly from the identities for integer r

$$\sum_{s=0}^{2N-1} \cos \tfrac{rs\pi}{N} = \tilde{\delta}_{r,0} \cdot 2N \quad \text{and} \quad \sum_{s=0}^{2N-1} \sin \tfrac{rs\pi}{N} = 0.$$

□

Next, we compute explicitly the interpolants for the jump functions f_ε. It turns out that the interpolant to f_ε is equal to the interpolant to f_0, shifted vertically by $\varepsilon/2$, plus a perturbation term that is completely independent of $\varepsilon \neq 0$.

LEMMA 2.2. *The trigonometric interpolants $L_N f_\varepsilon$ possess the following representations*

$$L_N f_0(t) = \tfrac{\pi}{2N} \sum_{k=1}^{N-1} \cot \tfrac{k\pi}{2N} \sin kt,$$

and for $0 < \varepsilon < \frac{\pi}{N}$

$$\begin{aligned} L_N f_\varepsilon(t) &= L_N f_0(t) - \tfrac{\pi}{2}\varphi_N(t) + \tfrac{\varepsilon}{2} \\ &= \tfrac{\varepsilon}{2} - \tfrac{\pi}{4N}\Big(1 + {\sum_{|k|=1}^{N}}{}' \Big(1 + i\cot\tfrac{k\pi}{2N}\Big)e^{ikt}\Big). \end{aligned}$$

Here $\sum'$ *means that the terms for* $|k| = N$ *have to be multiplied by* $1/2$.

Proof. For arbitrary $0 \le \varepsilon < \frac{\pi}{N}$ we obtain

$$L_N f_\varepsilon(t) = \tfrac{1}{2N}\sum_{s=0}^{2N-1}\sum_{\ell=1}^{\infty}\tfrac{\sin\ell(\frac{s\pi}{N}-\varepsilon)}{\ell}\Big(1 + 2{\sum_{k=1}^{N}}{}'\cos k(t - \tfrac{s\pi}{N})\Big).$$

Now we simplify for $0 < k \le N$ using Lemma 2.1

$$\begin{aligned} &\sum_{s=0}^{2N-1}\sum_{\ell=1}^{\infty}\tfrac{\sin\ell(\frac{s\pi}{N}-\varepsilon)}{\ell}\cos k(t - \tfrac{s\pi}{N}) = \\ &= \sum_{\ell=1}^{\infty}\tfrac{1}{\ell}\sum_{s=0}^{2N-1}\Big(\sin\tfrac{\ell s\pi}{N}\cos\ell\varepsilon\cos kt\cos\tfrac{ks\pi}{N} + \sin\tfrac{\ell s\pi}{N}\cos\ell\varepsilon\sin kt\sin\tfrac{ks\pi}{N} \\ &\qquad - \cos\tfrac{\ell s\pi}{N}\sin\ell\varepsilon\cos kt\cos\tfrac{ks\pi}{N} - \cos\tfrac{\ell s\pi}{N}\sin\ell\varepsilon\ddot{o}\sin kt\sin\tfrac{ks\pi}{N}\Big) \\ &= \sin kt\sum_{\ell=1}^{\infty}\tfrac{\cos\ell\varepsilon}{\ell}N(\tilde{\delta}_{\ell,k} - \tilde{\delta}_{\ell,-k}) - \cos kt\sum_{\ell=1}^{\infty}\tfrac{\sin\ell\varepsilon}{\ell}N(\tilde{\delta}_{\ell,k} + \tilde{\delta}_{\ell,-k}) \\ &= N\sin kt\Big(\tfrac{\cos k\varepsilon}{k} + \sum_{L=1}^{\infty}(\tfrac{\cos(k+2NL)\varepsilon}{k+2NL} - \tfrac{\cos(-k+2NL)\varepsilon}{-k+2NL})\Big) \\ &\quad -N\cos kt\Big(\tfrac{\sin k\varepsilon}{k} + \sum_{L=1}^{\infty}(\tfrac{\sin(k+2NL)\varepsilon}{k+2NL} + \tfrac{\sin(-k+2NL)\varepsilon}{-k+2NL})\Big) \\ &= \tfrac{1}{2}\sin kt\sum_{L=-\infty}^{\infty}\frac{\cos\big((\frac{k}{2N}+L)(2N\varepsilon)\big)}{\frac{k}{2N}+L} - \tfrac{1}{2}\cos kt\sum_{L=-\infty}^{\infty}\frac{\sin\big((\frac{k}{2N}+L)(2N\varepsilon)\big)}{\frac{k}{2N}+L} \\ &= \tfrac{\pi}{2}\sin kt\cdot\cot\tfrac{k\pi}{2N} - \begin{cases} 0, & \text{if } \quad \varepsilon = 0, \\ \frac{\pi}{2}\cos kt, & \text{if } \; 0 < 2N\varepsilon < 2\pi. \end{cases} \end{aligned}$$

For the series representation yielding the last equality, compare [6, pp. 71,73]. In the case $k = 0$ we write

$$\begin{aligned}
\sum_{s=0}^{2N-1}\sum_{\ell=1}^{\infty}\frac{\sin\ell(\frac{s\pi}{N}-\varepsilon)}{\ell} &= \sum_{\ell=1}^{\infty}\frac{1}{\ell}\sum_{s=0}^{2N-1}\Big(\sin\frac{\ell s\pi}{N}\cos\ell\varepsilon-\cos\frac{\ell s\pi}{N}\cdot\sin\ell\varepsilon\Big)\\
&= -2N\sum_{\ell=1}^{\infty}\frac{\sin\ell\varepsilon}{l}\tilde{\delta}_{\ell,0}\\
&= -2N\sum_{L=1}^{\infty}\frac{\sin 2NL\varepsilon}{2NL}\\
&= -\sum_{L=1}^{\infty}\frac{\sin 2NL\varepsilon}{L}\\
&= \begin{cases} 0, & \text{if} \quad \varepsilon=0,\\ \frac{2N\varepsilon-\pi}{2}, & \text{if} \quad 0<\varepsilon<\frac{\pi}{N}.\end{cases}
\end{aligned}$$

Summing up k from 0 to N we obtain the assertions of Lemma 2.2. □

The different behaviour of the Lagrange interpolants is illustrated by the following figures.

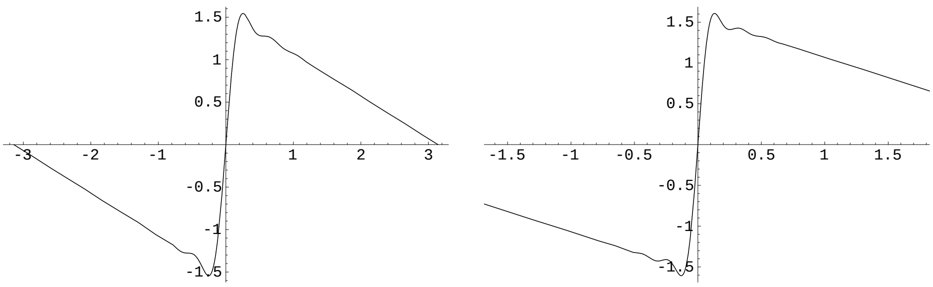

Fig. 1. Left: $L_{16}f_0$, Right: $L_{32}f_0$.

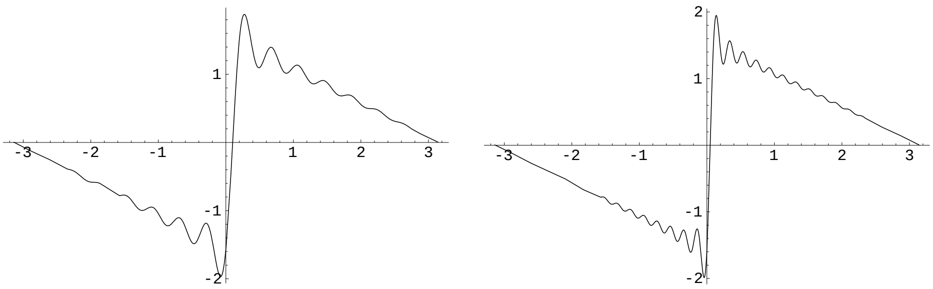

Fig. 2. Left: $L_{16}f_{0.01}$, Right: $L_{32}f_{0.01}$.

3. THE ARCLENGTH OF THE INTERPOLANT

For the arclengths of the jump function interpolants we obtain the following result.

THEOREM 3.1. *The length of the graph* $\Gamma(L_N f_\varepsilon)$ *of the interpolant* $L_N f_\varepsilon$ *remains bounded iff* $\varepsilon = 0$, *i.e.,*

$$\text{length}\Big(\Gamma(L_N f_0)\Big) = \mathcal{O}(1), \quad N \to \infty,$$

while for $0 < \varepsilon < \frac{\pi}{N}$

$$\text{length}\Big(\Gamma(L_N f_\varepsilon)\Big) \sim \log N, \quad N \to \infty.$$

Proof. For $\varepsilon = 0$ we obtain by definition

$$\text{length}\Big(\Gamma(L_N f_0)\Big) = \int_{-\pi}^{\pi} \sqrt{1 + \Big((L_N f_0)'(t)\Big)^2}\, dt \le 2\pi + \|(L_N f_0)'\|_1. \tag{2}$$

Using the function

$$g_0(x) = \begin{cases} 1, & \text{if} \quad x = 0, \\ 0, & \text{if} \quad |x| > \frac{\pi}{2}, \\ x \cot x, & \text{otherwise,} \end{cases}$$

we can write

$$\begin{aligned} (L_N f_0)'(t) &= \sum_{k=1}^{N-1} g_0\left(\tfrac{k\pi}{2N}\right) \cdot \cos kt \\ &= \tfrac{1}{2} \sum_{k=-N}^{N} g_0\left(\tfrac{k\pi}{2N}\right) e^{ikt} - \tfrac{1}{2}, \end{aligned}$$

and we can estimate with the help of Poisson's summation formula (cf. [1, Lemma 1])

$$\left\| \sum_{k=-N}^{N} g_0\left(\tfrac{k\pi}{2N}\right) e^{ik\circ} \right\|_1 \le \|\hat{g}_0\|_{L^1(\mathbb{R})},$$

where $\hat{g}_0$ is the Fourier transform of g_0. Now we can use (cf. [1, Lemma 3]) that

$$\|\hat{g}_0\|_{L^1(\mathbb{R})} \le 4\sqrt{V(g_0') \cdot \|g_0\|_{L^1(\mathbb{R})}}\,. \tag{3}$$

Here it holds that $\|g_0\|_{L^1(\mathbb{R})} = \pi \ln 2$, while for the total variation of the derivative one obtains $V(g_0') = 2\pi$ and hence

$$\|(L_N f_0)'\|_1 \le \pi + 4\pi\sqrt{2 \ln 2}.$$

This proves the first part of the theorem.

Using (2) and the representation of $L_N f_\varepsilon$ from Lemma 2.2, we conclude

$$\text{length}(\Gamma(L_N f_\varepsilon)) \sim \|\tfrac{\pi}{2}(\varphi_N)'\|_1.$$

From Bernstein's inequality it follows easily that

$$\|\tfrac{\pi}{2}(\varphi_N)'\|_1 \le \tfrac{\pi N}{2}\|\varphi_N\|_1 \le 2\ln N + C.$$

On the other hand the lower bound for $\|(\varphi_N^M)'\|_1$ is derived analogously to the standard arguments for the Dirichlet kernel (cf. [6, p. 67]). □

For the convenience of the reader we include plots of g_0 and g_0' to illustrate the smoothness properties of g_0.

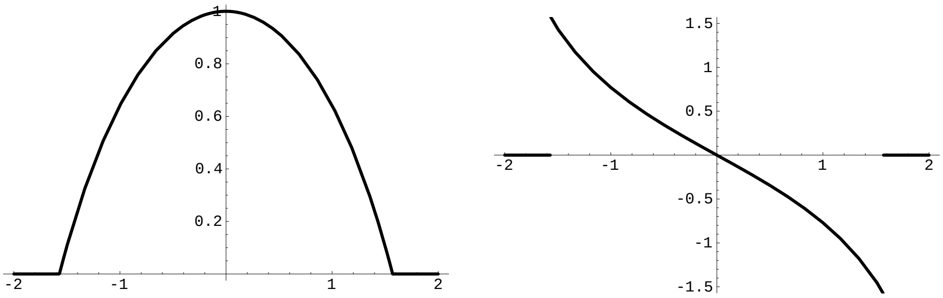

Fig. 3. Left: g_0 with $\|g_0\|_{L^1(\mathbb{R})} = \pi\ln 2$, Right: g_0' with $V(g_0') = 2\pi$.

Moreover, let us mention that for $\varepsilon > 0$

$$(L_N f_\varepsilon)'(t) = \tfrac{1}{2}\sum_{k=-N}^{N} g\left(\tfrac{k\pi}{2N}\right)e^{ikt} - \tfrac{1}{2}\,,$$

where

$$g(x) = \begin{cases} 1, & \text{if } \ x = 0, \\ 0, & \text{if } \ |x| > \frac{\pi}{2}, \\ x\cot x - ix & \text{otherwise.} \end{cases}$$

Then the real part of g is the smooth function g_0, whereas the imaginary part has jumps so that the estimate (3) does not hold.

In the following result we describe the behaviour of the arclength of the graph of the interpolant for smoother functions f.

THEOREM 3.2. *Let the 2π-periodic function f be sufficiently smooth in the sense that*

$$f' \in L_{2\pi}^p$$

for a certain $p > 1$. Then

$$\lim_{N\to\infty} \operatorname{length}(\Gamma(L_N f)) = \operatorname{length}(\Gamma(f)). \tag{4}$$

Proof. Following the ideas of Strichartz [5, Proposition 2] we estimate

$$|\operatorname{length}(\Gamma(f)) - \operatorname{length}(\Gamma(L_N f))| \le \|(f - L_N f)'\|_1.$$

In the next steps we need a mean of the Fourier sum which approximates in the order of best approximation for all $L^p_{2\pi}$-spaces and reproduces polynomials. For that reason we choose the de la Vallée Poussin mean

$$\sigma^N_{2N} f(t) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t-u)\Big(1 + 2\sum_{k=1}^{N} \cos ku + \sum_{k=N+1}^{3N-1} \tfrac{3N-k}{2N} \cos ku\Big) \mathrm{d}u.$$

We obtain by Bernstein's inequality

$$\begin{aligned}
\|(f - L_N f)'\|_1 &\le \|(f - \sigma^N_{2N} f)'\|_1 + \|(\sigma^N_{2N}(f - L_N f))'\|_1 \\
&\le cE_N(f', L^1_{2\pi}) + 2N\|\sigma^N_{2N}(f - L_N f)\|_1 \\
&\le cE_N(f', L^1_{2\pi}) + cN\|f - L_N f\|_1 \\
&\le cE_N(f', L^p_{2\pi}) + cN\|f - L_N f\|_p \\
&\le cE_N(f', L^p_{2\pi}), \qquad (5)
\end{aligned}$$

where for the last inequality we have used a result on trigonometric Lagrange interpolation proved in [4]. As $E_N(f', L^p_{2\pi})$ tends to zero for $f' \in L^p_{2\pi}$, the theorem is proved. $\square$

Note that specific orders of convergence in (4) can be obtained from (5) for sufficiently smooth functions by using standard Jackson type arguments for trigonometric best approximation.

One can also achieve bounded arclength in the case $\varepsilon > 0$ by modifying the Lagrange interpolation process. If one interpolates in $2N$ points one can allow the interpolation polynomial to have a degree bigger than N. Let us write for $1 \le M \le N$ (cf. [3])

$$\varphi^M_N(t) = \tfrac{1}{2N}\Big(1 + 2\sum_{k=1}^{N} \cos kt + \sum_{k=N-M+1}^{N+M-1} \tfrac{N+M-k}{2M} \cos kt\Big)$$

and

$$L^M_N f(t) = \sum_{s=0}^{2N-1} f\left(\tfrac{s\pi}{N}\right) \varphi^M_N\left(t - \tfrac{s\pi}{N}\right).$$

Then $L^M_N f(\frac{k\pi}{N}) = f(\frac{k\pi}{N})$ for all integer k and $L^M_N f$ is a trigonometric polynomial of degree less than $N+M$. Note also that $\varphi_N = \varphi^1_N$ and $L_N f = L^1_N f$. The particular feature of these interpolation polynomials L^M_N is the boundedness of the kernels φ^M_N depending on the quotient N/M only. Using the well-known estimate

$$\|\varphi^M_N\|_1 \sim \tfrac{4}{\pi N} \ln \tfrac{2N}{M}$$

and the same methods of proof as above, we obtain the following result.

Theorem 3.3. *Let N/M be bounded. Then for arbitrary ε and arbitrary values of f_ε at the jump it holds that*

$$\operatorname{length}\Big(\Gamma(L_N^M f_\varepsilon)\Big) = \mathcal{O}(1),\ \ N \to \infty\,,$$

and for 2π-periodic absolutely continuous functions f, i.e., $f' \in L^1_{2\pi}$, it holds that

$$\lim_{N\to\infty} \operatorname{length}(\Gamma(L_N f)) = \operatorname{length}(\Gamma(f)).$$

REFERENCES

[1] Girgensohn, R. and Prestin, J., *Lebesgue constants for an orthogonal polynomial Schauder basis*, J. Comput. Anal. Appl., **2**, pp. 159–175, 2000.

[2] Prestin, J. and Quak, E., *Trigonometric interpolation and wavelet decompositions*, Numer. Algor., **9**, pp. 293–318, 1995.

[3] Prestin, J. and Selig, K., *Interpolatory and orthonormal trigonometric wavelets*, in: Signal and Image Representation in Combined Spaces (Eds. J. Zeevi, R. Coifman), Academic Press, pp. 201–255, 1998.

[4] Xu, Y. and Prestin, J., *Convergence rate for trigonometric interpolation of non-smooth functions*, J. Approx. Theory, **77**, pp. 113–122, 1994.

[5] Strichartz, R. S. *Gibbs' phenomenon and arclength*, J. Fourier Anal. Appl., **6**, pp. 533–536, 2000.

[6] Zygmund, A., *Trigonometric Series*, Cambridge University Press, Second Ed., 1959.

Received December 19, 2000.