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CHEBYSHEV'S PROBLEM

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Dedicated to Professor Haussmann on his 60th birthday

Abstract. The Chebyshev approximation problem is usually described as to find the polynomial (or the element of an Haar subspace) which uniformly best approximates a given continuous function. Most of the theoretical results forming the basis of this theory have not been explored by members of the St Petersburg Mathematical School, founded by P. L. Chebyshev himself. The present article briefly wants to explain why. We show that the interests of Chebyshev and his most narrow pupil, A. A. Markov sr. focussed on a more algebraic problem, solved only in 1960s by Meyman and required techniques far away from what mathematicians were able to deal with at that time.

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1. IMPROVING THE STEAM ENGINE

Approximation techniques like Taylor, Fourier—expansion or Least square—approximation had been introduced long time before Chebyshev made the first steps in mathematics, but they share the disadvantage not to control the approximation error on a whole interval, resp. a compact set.

As well-known[1], the starting point for seeking best approximating functions was the problem to improve the effectiveness of Watt's mechanism. Since this mechanism is not able to convert the straight movement of the steam engine into the circular motion of the wheels without loss, Chebyshev tried to minimize this error.

This led to the problem to minimize the deviation between the so-called Watt's Curve, an algebraic curve of sixth degree with one singularity and the straight line, that is, the zero function[2].

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[1]The first description of Chebyshev's life can be found in the 1895 necrologue by A. M. Lyapunov [19]. More detailed information on Chebyshev and the beginning of the uniform approximation problem can be found in [28], [24], [18], [25], [3] and [26] (historical order).

[2]For details see [26].

Chebyshev himself presented his first approach to the solution of this in the famous paper [7], which is in fact the scientific[3] report of his (probably[4]) first trip abroad in 1852.

Here he already states a more abstract problem, namely [7, p. 114]

> „Déterminer les modifications qu'on doit apporter dans la valeur approchée de fx, donnée par son développement suivant les puissances de $x - a$, quand on cherche à rendre *minimum* la limite de ses erreurs entre $x = a - h$ et $x = a + h$, h étant une quantité peu considérable."

This problem is in fact a two-dimensional approximation problem with the interval length as a second variable. In the concrete case of Watt's mechanism he wants to control the error within a small intervall with maximum distance $\pm|h|$ from the singularity. In this approach the introduction of this variable is quite natural, since the interval length is crucial for Taylor expansion.

He is not able to solve this problem completely, and so his first approach is to find a best approximating solution with order h^{n+1}. He so can turn over to the problem of determining the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{1}$$

which least differs from zero in the interval $[-1, 1]$.

This is, as we say nowadays, the *Chebyshev polynomial* of the first kind. Chebyshev solves the problem by using characteristic equations for the $2n+4$ deviation points, which are a direct consequence from the necessary condition we know from the alternation theorem, if we neglect the signs. In [7] Chebyshev presents this condition as a "well-known" fact[5].

Using this solution he gives an algorithm which computes an approximative solution for the above cited general problem. He successively enlarges the approximation order, but is not able to give an exact solution for higher orders than $n + 1$, which coincides with the problem of determining polynomials of degree $m \geq n + 1$ deviating least possible from zero with $m - n$ given leading coefficients.

2. A GENERAL SETTING

Chebyshev's most important contribution to the theory of best uniform approximation is the ingenious work [11], where he investigates a more general setting, finds best approximating functions in three special cases and proves

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[3] The official report is [6].

[4] Compare the discussion in [2] and [26].

[5] Indeed, Chebyshev presents it with the words "comme on le sait" ([7, p. 154]). It is unknown, why he has this opinion. Since he will prove this condition later (in [11], neglecting the signs of the error function), we may assume that nevertheless he found the proof himself (comp. [17] and [26]).

a large number of corollaries which mostly deal with the distribution of roots of algebraic equations. His results reach so far that he is in fact able to prove Runge's theorem on the convergence of the Lagrange interpolation process for analytical functions by choosing Chebyshev nodes[6].

The setting is quite general. Chebyshev now considers a function

$$F : [-h, h] \times \mathbb{R}^n \to \mathbb{R}$$

depending on the variable x and n parameters $p_1, \ldots, p_n$, for example the coeffients of the approximating polynomial, which deviates least possible from zero. The only assumption on F is its differentiability both in all p_i (explicitly) and x (implicitly).

From the modern theoretical point of view the most important result is the following theorem:

THEOREM 1. *$F : [-h, h] \times \mathbb{R}^n \to \mathbb{R}$ is not the best approximation of the zero function, if the system of linear equations*

$$\begin{array}{rcl} N_1 \dfrac{\partial F}{\partial p_1}(x_1) + N_2 \dfrac{\partial F}{\partial p_2}(x_1) + \ldots + N_n \dfrac{\partial F}{\partial p_n}(x_1) & = & F(x_1) \\ N_1 \dfrac{\partial F}{\partial p_1}(x_2) + N_2 \dfrac{\partial F}{\partial p_2}(x_2) + \ldots + N_n \dfrac{\partial F}{\partial p_n}(x_2) & = & F(x_2) \\ \vdots & \vdots & \vdots \\ N_1 \dfrac{\partial F}{\partial p_1}(x_\mu) + N_2 \dfrac{\partial F}{\partial p_2}(x_\mu) + \ldots + N_n \dfrac{\partial F}{\partial p_n}(x_\mu) & = & F(x_\mu) \end{array} \tag{2}$$

has a non-trivial solution $(N_1, \ldots, N_n) \in \mathbb{R}^n$. x_i, $i = 1, \ldots, \mu$, are the (only) deviation points $F(., p_1, \ldots, p_n)$.

This theorem gives a lower limit for the number of deviation points of the error function and generalizes the fact which was cited before as "well-known". So it gives the most significant property of best approximating functions like polynomials, rational functions and other approximating families. However, this theorem remains the most general result in uniform approximation theory, not only of Chebyshev himself, but of the whole school founded by Chebyshev. Furthermore, direct consequences like the alternation condition and the unicity of the best uniform approximation, which hold for polynomials and rational functions, are even not mentioned by Chebyshev, although he gave all arguments to prove them easily.

This can only be understood by investigating the concrete aim of Chebyshev's research. The problem of determining the best possible setting for a mechanism has not only been the starting point for his research in approximation theory, but should dominate all his research in this subject for all his

[6]For details see [17] and [26].

life. The amount of his results in practical and theoretical mechanics exceeds all his other works, although mathematicians naturally appreciate and mention the milestones he set in his mathematical papers[7]. All these problems were in fact extremal problems of determining polynomials deviating least possible from zero under given side-conditions, like

(1) fixed coefficients (as above),
(2) monotonicity or
(3) interpolatory conditions.

Theoretically these problems have no connection at all. In solving them Chebyshev uses a wide area of results developed mostly by himself. Besides the uniform approximation these areas include the theory of orthogonal polynomials in $L_2(\rho)$ and reach back to his early research on integration theory[8].

It seems that here mathematics played purely the role to give *helpful tools,* for Chebyshev their results had no specific theoretical interest.

In his programmatic speech held at the St Petersburg University, Chebyshev explicitly insists on this role of the mathematical sciences:

> „The approximation of theory and practice gets the best results, and not only the practice takes the profit: the sciences develop themselves under the influence of the practice [...] If theory gets much from new applications of an old method [...], so it even more gains from the exploration of new methods, and in this case science finds for itself a real leader in the practice."

And indeed, his work on approximation theory focused on solving problems, concretely: *results without direct practical applications in mechanism theory were not interesting for him.*

3. WHAT HAVE CHEBYSHEV'S FUNCTIONS BEEN REALLY ABOUT?

In his first paper on uniform approximation theory he formulates the problem of approximating a function given by its Taylor series. In the second paper he speaks only about functions. In practical applications this difference does not occur, since all mechanic behaviour he deals with can be expressed by analytic functions.

Although he is familiar with Euler's concept of functions, he shows no deep interest in developing function theory or discussing for which functions his

[7]This is a pure mathematical point of view. Engineers point out that his results in mechanism theory could not be reached for at least fifty years, compare for example Tokarenko's contribution about Chebyshev's unique pupil in this subject [27].

[8]On Least-Squares-Approximation there are several contributions by Chebyshev beginning with [8], integration theory based on theorems of Abel-type was the subject of his lectureship thesis [5].

results may hold. His papers show that necessary restrictions like the differentiability of $F(., p_1, \ldots, p_n)$ in 1 are not mentioned explicitly or at most mentioned, when necessary, not when the function is introduced.

In his lectures he disparagingly speaks about conceptional discussions in analysis like infinitely small quantities. Ermolaeva cites from his lectures on probability theory [16]:

> "But philosophying about what an infinitely small number is, does not lead to anything. There may be only two conclusions: either to see that proofs using infinitely small numbers are not strong—and then we had to reject them, or to verify their strength."

Going back to his main work on uniform approximation theory [11], we already do not wonder how he himself defines as his aim which this work is subordinated a little bit disappointing from a modern point of view. In the paper [10] which gives a brief introduction to the voluminous paper [11], he writes:

> "For a given approximated expression of $f(x)$, usually as a polynomial or a rational expression, we have to determine the changes of the coeffcients, if the absolute error between $x = a - h$ und $x = a + h$ gets possibly small, where h is a possibly small number."

This is absolutely the same problem already defined in his first paper [7].

He himself could not reach this aim and concentrated on new problems connected with his beloved mechanisms. So he passed further steps to his pupils.

4. EXTREMAL PROBLEMS OF ZOLOTAREV AND THE MARKOV BROTHERS

The next steps on Chebyshev's algebraic approximation problems were made by E. I. Zolotarev (1847–1878), A. A. Markov sr. (1856–1922) and V. A. Markov (1871–1897).

A. A. Markov [20] investigated the problem of determining the polynomial p of n-th degree deviating the least possible from zero which suffices the linear side-condition

$$p'(x) = 1$$

for a given x. His brother generalized this problem to any linear side-condition [22]. The famous results from these works are the inequalities they proved, an upper limit for the first and the k-th derivative of a polynomial of n-th degree, respectively in terms of the maximum value of the polynomial.

Even closer to Chebyshev's problem was E. I. Zolotarev who realized the second step of his general question. In [29] he found the polynomial

$$x^n - \sigma x^{n-1} + \alpha_{n-2}x^{n-2} + \cdots + \alpha_0$$

deviating the least possible from zero. That is Chebyshev's problem for the leading two coefficients given. The solution of this problem required even more theoretical and technical skills as the original question. Zolotarev used many of techniques from the theory of elliptic functions and had to discriminate several cases, since the problem's setting allowed more varieties of the distribution of the zeros of the possible solution than the first problem. A detailed analysis of Zolotarev's solution has been made by Carlson and Todd [4].

5. TOO DIFFICULT TO REACH

As a whitness for the fact that indeed this problem was the main interest of Chebyshev's pupils, too, we can state A. A. Markov sr. In 1906 he managed to give a first summary of early St Petersburg results in approximation in his "Lectures on Functions Deviating the Least Possible from Zero". He there introduces Chebyshev's theorem in a little more general setting, nearly proves the alternation theorem[9], but after this general result turns to the above cited extremal problems.

Explicitly he does not cite Zolotarev's solution because it is "too complicated for practice", and calculates it only for a simple example.

About the general problem having n given coefficients he writes:

> To find the exact values of this coefficients for any given n and for any function f, satisfying the given conditions[10] is an extremely difficult problem, and we are far away from its solution."

Exactly this was the problem which hindered St Petersburg mathematicians in developing further results in approximation theory: their problem apparently turned out to be too difficult, results from abroad absolutely did not influence them[11].

Approximation theory from that time on followed another direction—questions of quantitative nature became more important, analytical problems replaced the algebraic questions of Chebyshev's school.

So there was not the attention paid for the next steps as it should be from a historical point of view.

Akhiezer found in 1928 the polynomial deviating the least possible from zero for the first three coefficients given and it was only in 1960 when Meyman could solve Chebyshev's problem for any number of given coefficients.

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[9]In the case of a finite number of deviating points.
[10]That is, deviating the least possible from zero.
[11]For a detailed discussion of this fact comp. [26].

REFERENCES

[1] Akhiezer, N. I., *Über die Functionen, die in gegebenen Intervallen am wenigsten von Null abweichen*, Izv. Kazansk. Soc. Phys. Math., **3**, 1928.

[2] Butzer, P. L., and Jongmans, F., *P. L. Chebyshev (1821–1894) and his contacts with western european scientists*, Historia Math., **16**, pp. 46–68, 1989.

[3] Butzer, P. L., and Jongmans, F., *P. L. Chebyshev (1821–1894). A guide to his life and work*, J. Approx. Theory, **96**, pp 111–138, 1999.

[4] Carlson, B. C. and Todd, J., *Zolotarev's first problem—the best approximation by polynomials of degree* $\le n-2$ *to* $x^n - n\sigma x^{n-1}$ *in* $[-1, 1]$, Aequationes Math., **26**, pp. 1–33, 1983.

[5] Chebyshev, P. L., *About integration with the help of logarithms*, Lectureship thesis, St Petersburg 1847, in: [15, Vol. 5, pp. 88–140] (Russian).

[6] Chebyshev, P. L., *Report by the extraordinary professor of the St Petersburg University Chebyshev about his trip abroad*, Zhurnal ministerstva narodnogo prosveshcheniya, ch. XXVIII, *otd.* IV, S. 2–14, in: [15, Vol. 5, pp. 246–255] (Russian).

[7] Chebyshev, P. L., *Théorie des mécanismes, connus sous le nom de parallélogrammes*, Mémoires présentes à l'Academie Impériale des Sciences de St. Pétersbourg par divers savants, **VII**, pp. 539–568, 1854.

[8] Chebyshev, P. L., *On continuous fractions*, Uch. zap. Imp. Akad. nauk, **III**, pp. 636–664, 1855, in: [15, Vol. II, pp. 103–126] (Russian).

[9] Chebyshev, P. L., *Drawing geographic maps*, Essay, written for the Ceremonial Act in the Imperial St Petersburg University Feb. 8^{th}, 1856, in: [15, Vol. 5, pp. 150–157] (Russian).

[10] Chebyshev, P. L., *Questions about minima connected with the approximative representation of functions (short remark)*, Bull. de la classe phys.-math. de l'Acad. Imp. des Sciences de St.-Pétersbourg, **XVI**, S. 145–149, 1857, in: [15, Vol. 2, S. 146–150] (Russian).

[11] Chebyshev, P. L., *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions*, Mém. de l'Acad. de St. Pétersbourg, série **VI**, t. VII, pp. 199–291, 1859.

[12] Chebyshev, P. L., *About functions deviating least possible from zero*, Enclosure to volume XXII of zap. Akad. Nauk N°1, St Petersburg 1873, in: [15, Vol. 2] (Russian).

[13] Chebyshev, P. L., *About functions deviating a little from zero with given values of the variable*, Mém. de l'Acad. de St. Pétersbourg, **LX**, St. Petersburg 1880, in: [15, Vol. 4] (Russian).

[14] Chebyshev, P. L., *Selected Mathematical Papers*, gostekhizdat tekhniko-teoreticheskoj literatury, Moscow–Leningrade, 1946 (Russian).

[15] Chebyshev, P. L., *Complete Collected Works*, Moscow–Leningrade, 1946–1951 (Russian).

[16] Ermolaeva, N. S., *The finite and infinte in the works of P. L. Chebyshev*, in: A. G. Barabasheva, (Ed.): Infiniteness in Mathematics: Philosophical and Historical Aspects, Yanus-K, Moscow, 1997 (Russian).

[17] Goncharov, V. L., *The theory of best approximation of functions*, in: Akademija nauk SSSR (Ed.), The Scientific Heritage of P. L. Chebyshev, vol. 1: Mathematics, Moscow–Leningrade 1945, pp. 122–172 (Russian). Translated in J. Approx. Theory, **106**, pp. 2–57, 2000.

[18] Gusak, A. A., *Prehistory and the beginning of the development of approximation theory*, istoriko-matematicheskie issledovaniya, **14**, pp. 289–348, 1961 (Russian).

[19] Lyapunov, A. M., *Pafnuty Lvovich Chebyshev*, soobshcheniya Kharkovskogo matematicheskogo obshchestva, 2-ya seriya, **IV**, N° 5 and 6, (1895), in: [14, S. 9–21] (Russian).

[20] Markov, A. A., *About a question of D. I. Mendeleev*, zap. akad. nauk, **62**, 1890 (Russian).
[21] Markov, A. A., *About functions deviating the least possible from zero*, composed after lectures of the member of the academy, A. A. Markov, St Petersburg 1906 (Russian).
[22] Markov, V. A., *About functions deviating the least possible from zero on a given interval*, tipografiya imp. akad. nauk, St Petersburg 1892 (Russian).
[23] Meyman, V. A., *Polynomials deviating least from zero with an arbitrary number of given coefficients*, Soviet Math. Dokl., **1**, pp. 72–75, 1960.
[24] Prudnikov, V. E., *P. L. Chebyshev—Scientist and Teacher*, gos. uchebno-pedagog. izd. min. prosveshch. RSFSR, Moscow 1950 (Russian).
[25] Prudnikov, V. E., *Pafnuty Lvovich Chebyshev. 1821–1894*, Nauka, Leningrade 1976 (Russian).
[26] Steffens, K.-G., *Zur Entstehungsgeschichte der Approximationstheorie unter besonderer Berücksichtigung der Beiträge der St. Petersburger Mathematischen Schule um P. L. Čebyšëv*, Dissertation, Duisburg, 2000.
[27] Tokarenko, A. M., *The contribution of N. B. Delone to the development of the mechanism theory*, in: A. N. Bogolyubov (Ed.): Mathematical Knowledge of Nature in its Development, Naukova dumka, Kyiv, pp. 153–156, 1987 (Russian).
[28] Vasilev, A. V. and Delone, N. B., *P. L. Tschebyschef*, Teubner, Leipzig, 1900.
[29] Zolotarev, E. I., *About a question on smallest quantities*, Lectureship dissertation, St Petersburg 1868, in: [30, vyp. 2, pp. 130–166] (Russian).
[30] Zolotarev, E. I., *Complete Collected Works*, izd. AN SSSR, Leningrade, 1932 (Russian).

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