REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 30 (2001) no. 2, pp. 127–134
ictp.acad.ro/jnaat

ON THE FLOW OF A VISCOUS THIN LAYER ON AN INCLINED PLANE DRIVEN BY A CONSTANT SURFACE TENSION GRADIENT

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Abstract. Steady flow of a thin layer (trickle, rivulet) of viscous fluid down an inclined surface is considered, via a thin-film approximation. The work extends the study by Duffy and Moffatt [7] of gravity-driven thin trickle of viscous fluid to include the effects of a surface tension gradient. It acts on the free surface of the layer. At the same time the work tries an alternative analysis to our traditional approaches exposed in [6] and the papers quoted there. Asymptotic and numerical results for several values of volume flux and surface tension gradients are carried out.

MSC 2000. 76D08, 76D45, 76M55.

Keywords. viscous flow, thin film approximation, surface tension gradient.

1. INTRODUCTION

Many viscous flow problems of practical importance involve small scale flows with free surface whose effects contribute significantly to the dynamics through superficial forces. One prototype problem that has received much attention is that of the draining of viscous layers down an inclined plane driven simultaneously by a surface tension gradient.

We consider the steady behavior of such a trickle of viscous liquid (which we take to be supplied at a prescribed volume flux) when the surface tension gradient is constant. We are particularly interested in *the study of Marangoni effect.* More specific, *we take into account a non-zero tangential (shear) stress and use a lubrication (thin-film) approximation.* This approximation linearizes the Navier-Stokes system and enables us to obtain the velocity field, the free-surface velocity, the pressure and the free-surface profile in closed form.

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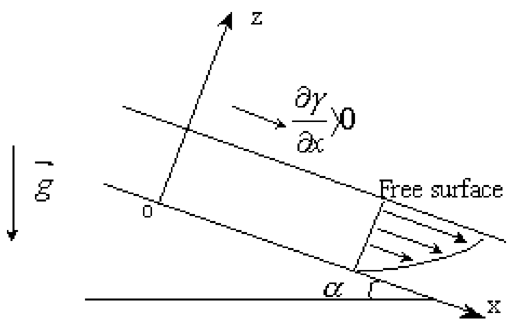
2. PROBLEM FORMULATION AND THE FLOW SOLUTION

Consider the flow of a uniform thin liquid layer (rivulet) down an inclined solid plane driven simultaneously by a surface tension gradient. This gradient acts on the free surface (the upper of the rivulet) and due to viscous forces can drag the fluid up or down on the incline. Suppose a Newtonian fluid, of constant density ρ and viscosity μ, which undergoes a steady rectilinear flow in the form of a filament, down a plane inclined at an angle α to the horizontal.

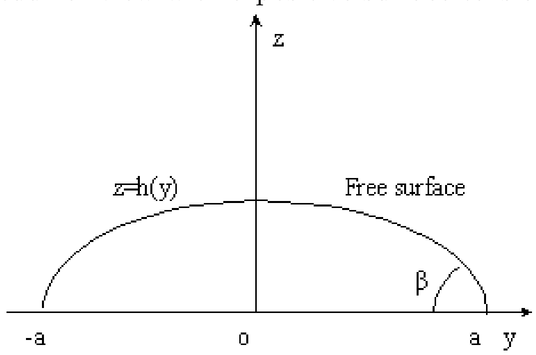
With respect to the Cartesian coordinates system $Oxyz$ as indicated in Fig.1, the velocity of this locally unidirectional flow will be of the form $\overrightarrow{u} = u\,(y,z)\,\overrightarrow{i}$ and the Navier-Stokes equations reduce to

$$\begin{cases} 0 = -p_x + \rho \cdot g \cdot \sin\alpha + \mu \cdot (u_{yy} + u_{zz}) \\ 0 = -p_y \\ 0 = -p_z - p \cdot g \cdot \cos\alpha \end{cases} \tag{1}$$

where the subscripts denote partial differentiation.



a) A longitudinal view with a positive surface tension gradient.



b) A transversal view (the transverse profile) of the flow.

Fig. 1: A trickle (rivulet) of viscous fluid, of width $2a$ and maximum depth $h_m := h(0)$ flowing down an inclined plane at an angle α to the horizontal.

In the thin-film (lubrication) theory, (see, for example, Acheson [1], Ockendon and Ockendon [11], or Gheorghiu [9]) these equations reduce to

$$(2) \qquad \begin{cases} 0 = -p_x + \rho \cdot g \cdot \sin\alpha + \mu \cdot u_{zz} \\ 0 = -p_y \\ 0 = -p_z - \rho \cdot g \cdot \cos\alpha \end{cases}$$

and are to be integrated subject to the boundary conditions

$$(3) \qquad \overrightarrow{u} = 0, \text{ on } z = 0$$

$$(4) \qquad \left. \begin{array}{l} p - p_a = -\overline{\gamma} \cdot (H + h)'' \\ \mu \cdot u_z = \tau \text{ (Levich–Aris boundary condition)} \end{array} \right\} \text{ on } z = (H + h)\,(y)\,.$$

Here $z = H\,(y)$ is the known equation of the transverse profile of the substrate (the bottom line) and $z = h\,(y)$ is the unknown equation of the transverse profile of the free surface. When the substrate is an inclined plane $H\,(y)$ becomes identically zero.

We impose the following contact conditions:

$$(5) \qquad \begin{cases} h = 0 \\ h' = \pm \tan\beta \end{cases}, \text{ at } y \pm a.$$

Here again $z = h\,(y)$ is the free-surface profile, p is the pressure in the liquid, p_a is the atmospheric pressure, $\overrightarrow{g}$ is the gravitational acceleration, $\overline{\gamma}$ is the reference value of surface tension, β is the contact angle at the three-phase contact line, $2a$ is the width of the layer (trickle), h_m is the maximum depth of the liquid, and τ is the constant shear stress acting on $z = h\,(y)$.

We consider β to be a prescribed constant such that $\beta < \pi/2$. A constant value of β means that any contact angle "hysteresis" is ignored. We appreciate this hypothesis as reasonable for these rivulets. On the physical nature of the shear stress τ we have to make the following important remark. The origin of τ could be very diverse. We are mainly interested when this stress comes from a local variation of surface tension, such that $\tau = \frac{\partial\gamma}{\partial x}$. Actually, γ may vary somewhat along x-axis and then the flow will not be truly unidirectional. However, the above approach may still be approximately correct if γ varies only slowly. A more sophisticated model that takes into account reactions, fluxes of surfactants on the free surface, surface diffusion and convection, change of metrics, etc., is available for example in [14]. To consider such models is beyond our computational capabilities. Consequently, we simply solve the differential system (2)-(5), using the strategy from Wilson and Duffy [15] and finally plug in $\frac{\partial\gamma}{\partial x}$ for the shear stress τ.

About the boundary condition (4) we observe that we have taken the surface curbature to be h''. The boundary condition (4) makes the difference between our study and those of Allen and Biggin [2], Duffy and Moffatt [7], Towel and Rothfeld[13], or Wilson and Duffy [15]. It means that the Marangoni effect is taken into account. Some mathematical aspects (existence, uniqueness, etc.) of this effect are addressed in Amick [3], Amick and Fraenkel [4] and Wagner [14].

With respect to the angle to the horizontal, we consider the following three cases:

i). $0 < \alpha < \dfrac{\pi}{2}$, ii) $\alpha = \dfrac{\pi}{2}$, iii) $\dfrac{\pi}{2} < \alpha < \pi$.

Then the solution read as follows:

the velocity

$$u(y,z) = \frac{\rho \cdot g \cdot \sin\alpha}{2\mu}\left(-z^2 + 2zh\right) + \frac{1}{\mu}\frac{\partial \gamma}{\partial x} \cdot z \tag{6}$$

the free-surface velocity, $u_s := u(y,h)$,

$$u_s = \frac{\rho \cdot g \cdot \sin\alpha}{2\mu} \cdot h^2 + \frac{1}{\mu}\frac{\partial \gamma}{\partial x} \cdot h, \tag{7}$$

and the pressure is

$$p(z) = p_a - \rho \cdot g \cdot z \cdot \cos\alpha + \tan\beta \sqrt{\rho \cdot g \cdot \overline{\gamma} \cdot |\cos\alpha|} \cdot \begin{cases} \coth B, & \alpha \in \left(0, \frac{\pi}{2}\right) \\ B^{-1}, & \alpha = \frac{\pi}{2} \\ \cot B, & \alpha \in \left(\frac{\pi}{2}, \pi\right). \end{cases} \tag{8}$$

The free-surface profile $z = h(y)$ is given by

$$\left(\frac{\rho \cdot g \cdot |\cos\alpha|}{\overline{\gamma}}\right)^{\frac{1}{2}} \frac{h}{\tan\beta} = \begin{cases} \frac{\cosh B - \cosh B\xi}{\sinh B}, & \alpha \in \left(0, \frac{\pi}{2}\right) \\ \frac{1}{2}B\left(1 - \xi^2\right), & \alpha = \frac{\pi}{2} \\ \frac{\cos B\xi - \cos B}{\sin B}, & \alpha \in \left(\frac{\pi}{2}, \pi\right) \end{cases} \tag{9}$$

where B is the Bond number for the flow, $B \neq 0$ for $\alpha \neq \pi/2$,

$$B = a\left(\frac{\rho \cdot g \cdot |\cos\alpha|}{\overline{\gamma}}\right)^{\frac{1}{2}} > 0 \text{ and } B = 0, \text{ for } \alpha = \frac{\pi}{2} \text{ and } \xi = \frac{y}{a},\ \xi \in [-1, 1]. \tag{10}$$

The maximum depth h_m of the liquid, $h_m := h\,(0)$ satisfies

$$(11)\qquad \left(\frac{p\cdot g\cdot|\cos\alpha|}{\overline{\gamma}}\right)^{\frac{1}{2}}\cdot\frac{h_m}{\tan\beta}=\begin{cases}\tanh\frac{1}{2}B, & \alpha\in\left(0,\frac{\pi}{2}\right)\\ \frac{1}{2}B, & \alpha=\frac{\pi}{2}\\ \tan\frac{1}{2}B, & \alpha\in\left(\frac{\pi}{2},\pi\right)\end{cases}$$

The scales of h_m and a in eqs. (10) and (11) differ essentially by the small factor $\tan\beta$ (and indeed in case ii) $h_m/a=\frac{1}{2}\tan\beta$). This reflects the fact that the depth of the layer is much less than its width. The solution is physically sensible only for $h\,(y)\geq 0$.

The volume flux of liquid running down the plane is

$$(12)\qquad Q=\int\limits_{-a}^{a}\int\limits_{a}^{h(y)}u\,dzdy=\frac{\rho\cdot g\cdot\sin\alpha}{3\mu}\int\limits_{-a}^{a}h^3\,(y)\,dy+\frac{1}{2\mu}\frac{\partial\gamma}{\partial x}\int\limits_{-a}^{a}h^2\,(y)\,dy.$$

It is convenient to introduce appropriate nondimensional variables defined by: $x^*=\dfrac{x}{a}$, $y^*=\dfrac{y}{a}$, $z^*=\dfrac{z}{a}h^*=\dfrac{h}{a}$, $\gamma^*=\dfrac{\gamma}{\overline{\gamma}}$, $u^*=\frac{u}{U}$, $p^*=\frac{p}{P}$ (where $P=\rho g a$ is the reference pressure), $Q^*=\frac{Q}{\overline{Q}}$, where $\overline{Q}=Ua^2$.

The Reynolds number is $\mathrm{Re}=\frac{UL}{\nu}=\frac{UL\rho}{\mu}$ and the Weber number is $We=\frac{\rho}{\rho LU^2}$, where we assume $\nu=\frac{\mu}{\rho}$, $\sigma=\overline{\gamma}$, $L=a$ and $U=\frac{\rho g a^2\sin\alpha}{\mu}$. Consequently, $\mathrm{Re}=\frac{Lg\rho^2a^2\sin\alpha}{\mu^2}$, $We=\frac{\rho\mu^2}{\rho^3Lg^2a^4\sin\alpha}$ and then $\mathrm{Re}\cdot We=\frac{\rho}{\rho g a^2}$. With the Bond number $Bo_1=\frac{\rho g L^2}{\sigma}$,which relates the gravitational forces g to the cappilarity, the nondimensional velocity of the fluid is given by

$$(13)\qquad u^*=\left(2z^*h^*-z^{*2}\right)+\frac{1}{Bo_1}\frac{\partial\gamma^*}{\partial x^*}z^*.$$

The nondimensional free surface velocity is

$$(14)\qquad u_s^*=\tfrac{1}{2}h^{*2}+\frac{1}{Bo_1\sin\alpha}\frac{\partial\gamma^*}{\partial x^*}h^*.$$

and the nondimensional pressure becomes

$$p^* = \begin{cases} p_a^* - |\cos\alpha| \cdot z^* + \frac{\tan\beta}{Bo_1} \cdot B \coth B, & \alpha \in \left(0, \frac{\pi}{2}\right) \\ p_a^* - |\cos\alpha| \cdot z^* + \frac{\tan\beta}{Bo_1} \cdot 1, & \alpha = \frac{\pi}{2} \\ p_a^* - |\cos\alpha| \cdot z^* + \frac{\tan\beta}{Bo_1} \cdot B \cot B, & \alpha \in \left(\frac{\pi}{2}, \pi\right). \end{cases}$$

The nondimensional version for the volume is given by

(15) $$Q^* = \frac{1}{3}\int_{-1}^{1} (h^*)^3\, dy^* + \frac{1}{2}\frac{1}{Bo_1 \sin\alpha}\frac{\partial\gamma^*}{\partial x^*}\int_{-1}^{1} (h^*)^2\, dy^*.$$

3. ASYMPTOTIC ANALYSIS

A) The limit $a \to 0$ $(B \to 0)$
We note that for all $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$,

$$\begin{aligned} h &\sim \tfrac{1}{2}a \cdot \left(1 - \xi^2\right) \cdot \tan\beta, \\ h_m &\sim \tfrac{1}{2}a \cdot \tan\beta, \text{ and} \\ \overline{Q} &\sim \frac{12B^4}{35} + \frac{\partial\gamma^*}{\partial x^*}\frac{1}{\tan\beta \cdot |\tan\alpha|}\frac{\rho \cdot g\,|\cos\alpha|}{\overline{\gamma}}\frac{6}{5}B^2, \text{ as } B \to 0. \end{aligned}$$

B). The limit $\beta \to 0$
We have respectively

$$\begin{aligned} h &\sim \beta \cdot \left(\frac{\overline{\gamma}}{\rho \cdot g \cdot |\cos\alpha|}\right)^{1/2} \frac{\cosh B - \cosh B\xi}{\sinh B}, \text{ for } \alpha \in \left(0, \tfrac{\pi}{2}\right), \\ h &\sim \beta \cdot \left(\frac{\overline{\gamma}}{\rho \cdot g \cdot |\cos\alpha|}\right)^{1/2} \cdot \tfrac{1}{2}B \cdot \left(1 - \xi^2\right), \text{ for } \alpha = \tfrac{\pi}{2} \\ h &\sim \beta \cdot \left(\frac{\overline{\gamma}}{\rho \cdot g \cdot |\cos\alpha|}\right)^{1/2} \cdot \frac{\cos B\xi - \cos B}{\sin B}, \text{ for } \alpha \in \left(\tfrac{\pi}{2}, \pi\right), \\ h_m &\sim \beta \cdot \left(\frac{\overline{\gamma}}{\rho \cdot g \cdot |\cos\alpha|}\right)^{1/2} \cdot \begin{cases} \tanh \frac{1}{2}B, & \alpha \in \left(0, \frac{\pi}{2}\right) \\ \frac{1}{2}B, & \alpha = \frac{\pi}{2} \\ \tan \frac{1}{2}B, & \alpha \in \left(\frac{\pi}{2}, \pi\right) \end{cases} \end{aligned}$$

and

$$\overline{Q} \sim \left(15B\coth^3 B - 15\coth^2 B - 9B\coth B + 4\right) + \frac{9}{2}\frac{\partial\gamma^*}{\partial x^*}\frac{1}{\beta\cdot|\tan\alpha|}\cdot$$
$$\cdot B^{-1}\left(3B\coth^2 B - 3\coth B - B\right) \text{ for } \alpha\in\left(0,\tfrac{\pi}{2}\right) \text{ as } \beta\to 0.$$

C). The limit $B\to\pi^-$
In case iii) we have

$$a \sim \pi\cdot\left(\frac{\gamma}{\rho\cdot g\cdot|\cos\alpha|}\right)^{1/2},$$
$$h \sim \tan\beta\cdot\left(\frac{\overline{\gamma}}{\rho\cdot g\cdot|\cos\alpha|}\right)^{1/2}\cdot\frac{\cos B\xi+1}{\pi-B},$$
$$h_m \sim \tan\beta\cdot\left(\frac{\overline{\gamma}}{\rho\cdot g\cdot|\cos\alpha|}\right)^{1/2}\cdot\frac{2}{\pi-B} \text{ and}$$
$$\overline{Q} \sim 15\pi\cdot(\pi-B)^{-3} + \frac{\partial\gamma^*}{\partial x^*}\frac{1}{\tan\beta\cdot\tan\alpha}27\pi\cdot B^{-1}, \text{ as } B\to\pi^-.$$

D). The limit $B\to\infty$ $(\alpha\to 0^+)$
In case i) we have

$$h \sim \tan\beta\cdot\left(\frac{\overline{\gamma}}{\rho\cdot g\cdot|\cos\alpha|}\right)^{1/2},$$
$$h_m \sim \tan\beta\cdot\left(\frac{\overline{\gamma}}{\rho\cdot g\cdot|\cos\alpha|}\right)^{1/2} \text{ and}$$
$$\overline{Q} \sim 6B + \frac{9}{\tan\beta\cdot|\tan\alpha|}\frac{\partial\gamma^*}{\partial x^*}.$$

CONCLUSIONS

We have to remark at the end of this paper that, when *we take into account surface tension gradients* which compete with gravity, they give rise to some specific terms in the expressions of velocity (6) or (13), free surface velocity (7) or (14) and the volume flux of the fluid (12) or (15)

They are respectively $\frac{1}{\mu}\frac{\partial\gamma}{\partial x}\cdot z$, $\frac{1}{\mu}\frac{\partial\gamma}{\partial x}\cdot h$, $\frac{1}{2\mu}\frac{\partial\gamma}{\partial x}\int\limits_{-a}^{a} h^2(y)\,dy$ and they have the same sign as $\frac{\partial\gamma}{\partial x}$. This fact is very plausible from physical point of view.

We have also to observe that the extra term in the expression of Q persist in all asymptotic expansions.

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Received July 7, 2000.