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APPROXIMATION OPERATORS CONSTRUCTED BY MEANS OF SHEFFER SEQUENCES

MARIA CRĂCIUN

Abstract. In this paper we introduce a class of positive linear operators by using the "umbral calculus", and we study some approximation properties of it. Let Q be a delta operator, and S an invertible shift invariant operator. For $f \in C[0,1]$ we define

$$(L_n^{Q,S} f)(x) = \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \tbinom{n}{k} p_k(x) s_{n-k}(1-x) f\left(\tfrac{k}{n}\right),$$

where $(p_n)_{n\geq 0}$ is a binomial sequence which is the basic sequence for Q, and $(s_n)_{n\geq 0}$ is a Sheffer set, $s_n = S^{-1} p_n$. These operators generalize the binomial operators of T. Popoviciu.

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1. INTRODUCTION

Let P be the linear space of all polynomials with real coefficients, and P_n the linear space of all polynomials of degree at most n.

We will consider some linear operators defined on P. We will denote by I the identity and by D the derivative. The shift operator $E^a : P \to P$ is defined by $E^a p(x) = p(x+a)$.

A linear operator T which commutes with all shift operators is called a *shift invariant operator*. In symbols, $E^a T = T E^a$, for all real a.

Let us remind that if T_1 and T_2 are shift invariant operators, then $T_1 T_2 = T_2 T_1$.

DEFINITION 1. *A shift invariant operator for which $Qx = const \neq 0$ is called a delta operator.*

By a *polynomial sequence* we shall denote a sequence of polynomials $p_n(x)$, $n = 0, 1, 2, \ldots$ where $p_n(x)$ is of degree exactly n for all n.

"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68–1, 3400 Cluj–Napoca, Romania, e-mail: craciun@ictp-acad.math.ubbcluj.ro.

A sequence of *binomial type* is a polynomial sequence $(p_n)_{n\geq 0}$ with $p_0(x) = 1$ and satisfying the identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y),$$

for all x, y and $n = 0, 1, 2, \ldots$.

Definition 2. *Let Q be a delta operator and $(p_n(x))_{n\geq 0}$ a polynomial sequence. If*

i) $p_0(x) = 1$,
ii) $p_n(0) = 0,\ n = 1, 2, \ldots$,
iii) $Qp_n = np_{n-1},\ n = 1, 2, \ldots$,

then (p_n) is called the sequence of basic polynomials for Q.

Proposition 1. [8].

i) *Every delta operator has a unique sequence of basic polynomials.*
ii) *If $p_n(x)$ is a basic sequence for some delta operator Q, then it is binomial.*
iii) *If $p_n(x)$ is a binomial sequence, then it is a basic sequence for some delta operator Q.*

Let X be the *multiplication operator* defined as $(Xp)(x) = xp(x)$ for every polynomial p.

For any operator T defined on P, the operator $T' = TX - XT$ is called the *Pincherle derivative* of the operator T.

Proposition 2. [8].

i) *If T is a shift invariant operator, then its Pincherle derivative is also a shift invariant operator.*
ii) *If Q is a delta operator, then its Pincherle derivative Q' is an invertible operator.*

Proposition 3. [8], [11]. *If $(p_n(x))_{n\geq 0}$ is a sequence of basic polynomials for the delta operator Q then*

i) $p_n(x) = X(Q')^{-1} p_{n-1}(x),\ n = 1, 2, \ldots$,
ii) $p_n(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-1-k}(x) p'_{k+1}(0),\ n = 1, 2, \ldots$

Definition 3. *A polynomial sequence $(s_n(x))_{n\geq 0}$ is called a Sheffer set relative to the delta operator Q if:*

i) $s_0(x) = const \neq 0$
ii) $Qs_n = ns_{n-1},\ n = 1, 2, \ldots$

An Appel set is a Sheffer set relative to the derivative D.

PROPOSITION 4. [11]. *Let Q be a delta operator with basic polynomial set $(p_n(x))_{n\geq 0}$ and $(s_n(x))_{n\geq 0}$ a polynomial sequence. The next statements are equivalent:*

i) *$s_n(x)$ is a Sheffer set relative to Q.*
ii) *There exists an invertible shift invariant operator S such that $s_n(x) = S^{-1}p_n(x)$.*
iii) *For all $x, y \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$, the following identity holds:*

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, s_{n-k}(y).$$

From the previous Proposition it results that the pair (Q, S) gives us a unique Sheffer set.

2. THE OPERATORS CONSTRUCTED BY MEANS OF SHEFFER POLYNOMIALS AND THEIR CONVERGENCE

In 1931 in [9] Tiberiu Popoviciu has used binomial sequences in order to construct some operators of the form

$$(L_n f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\tfrac{k}{n}\right) \tag{1}$$

where $f \in C[0,1]$ and $x \in [0,1]$. These operators are called *binomial operators.*

Such operators and their generalizations have been studied by the Romanian mathematicians as: D. D. Stancu, A. Lupaş, L. Lupaş, G. Moldovan, C. Manole, O. Agratini, A. Vernescu, and others.

Let Q be a delta operator and S an invertible shift invariant operator. Let $(p_n(x))_{n\geq 0}$ be the sequence of basic polynomials for Q, and $(s_n(x))_{n\geq 0}$ a Sheffer set relative to Q, $s_n = S^{-1}p_n$ with $s_n(1) \neq 0$ for any positive integer n.

In this note we want to study the operators $L_n^{Q,S} : C[0,1] \to C[0,1]$,

$$(L_n^{Q,S} f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right) \tag{2}$$

Because $p_k(0) = \delta_{k,0}$ (from the definition of basic polynomials), we have $(L_n^{Q,S} f)(0) = f(0)$.

In order to evaluate expression $(L_n^{Q,S} e_m)(x)$, where $e_m(x) = x^m$ we shall make use of C. Manole's method for binomial operators (see [5]) which we have adapted to our purposes.

Let us introduce the polynomials

$$S_m(x,y,n)=\sum_{k=0}^{n}\tbinom{n}{k}p_k(x)s_{n-k}(y)\left(\tfrac{k}{n}\right)^m \tag{3}$$

From Proposition 4 iii) we have $S_0(x,y,n)=s_n(x+y)$.

In the following we consider that x is the variable. Let us denote $\theta = X(Q')^{-1}$.

From Proposition 3 i) it results that $\theta p_k(x)=p_{k+1}(x)$ and consequently the linear operator θ is called the *shift operator* for the sequence $(p_n)_{n\geq 0}$ (see [10]). Therefore $\theta Q p_k(x)=\theta(kp_{k-1}(x))=kp_k(x)$; consequently k is an eigenvalue for the operator θQ, with its eigenvector $p_k(x)$. We have

$$(\theta Q)^m = k^m p_k(x) \tag{4}$$

for every positive integer m, and then

$$\begin{aligned} S_m(x,y,n) &= \tfrac{1}{n^m}(\theta Q)^m\sum_{k=0}^{n}\tbinom{n}{k}p_k(x)s_{n-k}(y)\\ &= \tfrac{1}{n^m}(\theta Q)^m S_0(x,y,n)=\tfrac{1}{n^m}(\theta Q)^m s_n(x+y). \end{aligned}$$

In this way we obtain

$$S_m(x,y,n)=\tfrac{1}{n^m}(\theta Q)^m E^y s_n(x) \tag{5}$$

Using the operational formula (see for instance [10])

$$(\theta Q)^m=\sum_{k=0}^{n}S(m,k)\theta^k Q^k,$$

where $S(m,k)=[0,1,\ldots,k;e_m]$ are the Stirling numbers of the second kind, relation (5) becomes:

$$S_m(x,y,n)=\tfrac{1}{n^m}\sum_{k=0}^{n}S(m,k)\theta^k Q^k E^y s_n(x). \tag{6}$$

Because Q is shift invariant and $Q^k s_n(x)=n(n-1)\ldots(n-k+1)s_{n-k}(x)=n^{[k]}s_{n-k}(x)$ we obtain

$$S_m(x,y,n)=\tfrac{1}{n^m}\sum_{k=0}^{n}\tbinom{n}{k}k!S(m,k)\theta^k E^y s_{n-k}(x),\ \forall m\in\mathbb{N}^* \tag{7}$$

THEOREM 1. *If $L_n^{Q,S}$ is the linear operator defined by* (2) *then*

$$
\begin{aligned}
(L_n^{Q,S}e_0)(x) &= e_0(x)\\
(L_n^{Q,S}e_1)(x) &= a_n e_1(x)\\
(L_n^{Q,S}e_2)(x) &= b_n x^2 + x(a_n - b_n - c_n),
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
a_n &= \frac{[(Q')^{-1}s_{n-1}](1)}{s_n(1)},\\
b_n &= \frac{n-1}{n}\frac{[(Q')^{-2}s_{n-2}](1)}{s_n(1)},\\
c_n &= \frac{n-1}{n}\frac{[(Q')^{-2}(S^{-1})'Ss_{n-2}](1)}{s_n(1)}
\end{aligned}
\tag{9}
$$

Proof. Using the notation (3) we can write

$$(L_n^{Q,S}e_m)(x) = S_m(x, 1-x, n)/s_n(1). \tag{10}$$

Because $S_0(x, 1-x, n) = s_n(1)$ we have $(L_n^{Q,S}e_0)(x) = e_0(x)$.
As we have

$$\theta E^y s_{n-1}(x) = X(Q')^{-1}E^y s_{n-1}(x) = XE^y(Q')^{-1}s_{n-1}(x), \tag{11}$$

we obtain from (7): $S_1(x, 1-x, n) = x[(Q')^{-1}s_{n-1}](1)$; consequently we get:

$$(L_n^{Q,S}e_1)(x) = \frac{[(Q')^{-1}s_{n-1}](1)}{s_n(1)}x.$$

Using the Pincherle derivative of the shift operator E^y

$$(E^y)' = yE^y = E^yX - XE^y \tag{12}$$

we can write

$$\theta E^y s_{n-k}(x) = E^yX(Q')^{-1}s_{n-k}(x) - yE^y(Q')^{-1}s_{n-k}(x)$$

Then

$$\theta^2 E^y s_{n-k}(x) = XE^y(Q')^{-1}E^yX(Q')^{-1}s_{n-k}(x) - yXE^y(Q')^{-2}s_{n-k}(x) \tag{13}$$

Because $s_{n-k} = S^{-1}p_{n-k}$, $XS^{-1} = S^{-1}X - (S^{-1})'$ (from the definition of Pincherle derivative) and $(Q')^{-1}p_{n-k}(x) = p_{n-k+1}(x)/x$ (from Proposition 3 i), we obtain

$$
\begin{aligned}
\theta^2 E^y s_{n-k}(x) =& XE^y(Q')^{-1}s_{n-k+1}(x) - XE^y(Q')^{-2}(S^{-1})'Ss_{n-k}(x)-\\
&- yXE^y(Q')^{-2}s_{n-k}(x).
\end{aligned}
\tag{14}
$$

Replacing (11) and (14) in (7) we can write

$$S_2(x,y,n) = xE^y(Q')^{-1}s_{n-1}(x) - \tfrac{n-1}{n}[xE^y(Q')^{-2}(S^{-1})'Ss_{n-2}(x) - yxE^y(Q')^{-2}s_{n-2}(x)]$$

From (10) and the previous relation one obtains expression $L_n^{Q,S}e_2$ from theorem's conclusion. □

LEMMA 1. *Let Q be a delta operator and S an invertible shift invariant operator. Let $(p_n(x))_{n\geq 0}$ be the sequence of basic polynomials for Q and $(s_n(x))_{n\geq 0}$ a Sheffer set relative to Q, $s_n = S^{-1}p_n$ with $s_n(1) \neq 0$ for any positive integer n. If $p_k'(0) \geq 0$ and $s_k(0) \geq 0$ for $n = 0, 1, 2, \ldots$ then the operator $L_n^{Q,S}$ defined by* (2) *is positive.*

Proof. If $p_k'(0) \geq 0$ using Proposition 3 ii), it is easy to prove by induction that $p_k(x) \geq 0$, $\forall\ k \in \mathbb{N}$ and $\forall x \in [0,1]$.

If we consider $x = 0$ in Proposition 4 iii) we obtain

$$s_n(x) = \sum_{k=0}^{n} \tbinom{n}{k} p_k(x)\, s_{n-k}(0)\,;$$

accordingly, for $s_k(0) \geq 0$ and $p_k(x) \geq 0$, $\forall\ k \in \mathbb{N}$, $\forall x \in [0,1]$, we have $s_k(x) \geq 0$, $\forall\ k \in \mathbb{N}$ and $\forall x \in [0,1]$. Therefore the operator $L_n^{Q,S}$ is positive. □

LEMMA 2. *If the operator $L_n^{Q,S}$ is positive, then $a_n \in [0,1]$, $b_n \leq 1$ and $0 \leq c_n \leq \min\{\frac{1-b_n}{2}, a_n - a_n^2\}, \forall n \in \mathbb{N}$, where a_n, b_n and c_n are defined by* (9).

Proof. Since $0 \leq e_1(t) \leq 1$, $\forall\ t \in [0,1]$ and the operator $L_n^{Q,S}$ is positive, we have $0 \leq (L_n^{Q,S}e_1)(x) \leq 1$, $\forall\ x \in [0,1]$, and as $(L_n^{Q,S}e_1)(x) = a_n x$, we get $a_n \in [0,1]$.

From $t(1-t) \geq 0$ it results that $(L_n^{Q,S}e_1)(x) - (L_n^{Q,S}e_2)(x) \geq 0$, which leads to $x(1-x)b_n + xc_n \geq 0, \forall\ x \in [0,1]$ and choosing $x = 1$, we get $c_n \geq 0$.

Since $t^2 - t + 1/4 \geq 0$, we obtain $(L_n^{Q,S}e_2)(x) - (L_n^{Q,S}e_1)(x) + (L_n^{Q,S}e_0)(x)/4 \geq 0, \forall\ x \in [0,1]$, relation equivalent to $x^2 b_n - xb_n - xc_n + 1/4 \geq 0, \forall x \in [0,1]$. If we consider $x = 1/2$, it results that $c_n \leq (1-b_n)/2$ and because $c_n \geq 0$ we get $b_n \leq 1$.

Finally, from the Schwarz's inequality,

$$[(L_n^{Q,S}e_1)(x)]^2 \leq (L_n^{Q,S}e_2)(x)(L_n^{Q,S}e_0)(x),$$

we have $a_n^2 x^2 \le b_n x^2 + x(a_n - b_n - c_n), \forall x \in [0,1]$. For $x = 1$ that implies $c_n \le a_n - a_n^2$. □

Theorem 2. *Let Q be a delta operator and S an invertible shift invariant operator. Let $(p_n(x))_{n\ge 0}$ be the sequence of basic polynomials for Q, with $p_n'(0) \ge 0, \forall n \in \mathbb{N}$, and $(s_n(x))_{n\ge 0}$ a Sheffer set relative to Q, $s_n = S^{-1}p_n$ with $s_n(1) \ne 0$ and $s_n(0) \ge 0, \forall n \in \mathbb{N}$. If $f \in C[0,1]$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$, where a_n and b_n are defined by (9), then the operator $L_n^{Q,S}$ converges to the function f, uniformly on the interval $[0,1]$.*

Proof. If $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty}(L_n^{Q,S} e_1)(x) = e_1(x)$. From Lemma 2, $c_n \le a_n - a_n^2$ so we have $\lim_{n\to\infty} c_n = 0$,and as $\lim_{n\to\infty} b_n = 1$, we get $\lim_{n\to\infty}(L_n^{Q,S} e_2)(x) = e_2(x)$. Therefore $\lim_{n\to\infty}(L_n^{Q,S} e_i)(x) = e_i(x)$ for $i = 0, 1, 2$ so we can use the convergence criterion of Bohman–Korokvin. □

3. REPRESENTATIONS OF THE OPERATOR $L_n^{Q,S}$

Theorem 3. *The operator $L_n^{Q,S}$ can be represented in the form*

$$(L_n^{Q,S} f)(x) = \sum_{k=0}^{n} \tfrac{k!}{n^k}\tbinom{n}{k}\left[0, \tfrac{1}{n}, \ldots, \tfrac{k}{n}; f\right] d_{k,n}(x), \tag{15}$$

where

$$d_{k,n}(x) = \tfrac{1}{s_n(1)}(\theta^k E^{1-x} s_{n-k})(x).$$

Moreover $L_n^{Q,S}(P_m) \subseteq P_m, \forall m \in \mathbb{N}$.

Proof. From the Newton interpolation formula we have

$$f\left(\tfrac{k}{n}\right) = \sum_{j=0}^{k} \tfrac{j!}{n^j}\tbinom{k}{j}\left[0, \tfrac{1}{n}, \ldots, \tfrac{j}{n}; f\right]$$

If we denote $w_{k,n}(x,y) = \binom{n}{k} p_k(x) s_{n-k}(y)$ then

$$\sum_{k=0}^{n} w_{k,n}(x,y) f\left(\tfrac{k}{n}\right) = \sum_{k=0}^{n} \tfrac{k!}{n^k}\left[0, \tfrac{1}{n}, \ldots, \tfrac{k}{n}; f\right] \sum_{j=k}^{n} \tbinom{j}{k} w_{j,n}(x,y).$$

But

$$\begin{aligned}\sum_{j=k}^{n}\tbinom{j}{k}w_{j,n}(x,y) &= \tbinom{n}{k}\sum_{j=k}^{n}\tbinom{n-k}{j-k}p_j(x)s_{n-j}(y)\\ &= \tbinom{n}{k}\sum_{j=0}^{n-k}\tbinom{n-k}{j}p_{j+k}(x)s_{n-k-j}(y)\\ &= \tbinom{n}{k}\theta^k\sum_{j=0}^{n-k}\tbinom{n-k}{j}p_j(x)s_{n-k-j}(y)\\ &= \tbinom{n}{k}\theta^k E^y s_{n-k}(x)\end{aligned}$$

Since $(L_n^{Q,S}f)(x)=\frac{1}{s_n(1)}\sum_{k=0}^{n}w_{k,n}(x,1-x)f(\frac{k}{n})$ we obtain (15).

In order to show that $L_n^{Q,S}(P_m)\subseteq P_m$ we shall prove that $deg(d_{k,n}(x))=k$.

We remind that if (p_n) is a basic sequence for $Q=q(D)$ and $h(t)$ is the compositional inverse of $q(t)$, then the generating function for (p_n) is

$$\sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}=e^{xh(t)} \tag{16}$$

and if $s_n=S^{-1}p_n$, with $S=s(D)$ then

$$\sum_{k=0}^{\infty}s_k(x)\tfrac{t^k}{k!}=\tfrac{1}{s(h(t))}e^{xh(t)} \tag{17}$$

If we differentiate the relation (16) m times with respect to t, we get

$$\sum_{k=0}^{\infty}p_{k+m}(x)\tfrac{t^k}{k!}=\tfrac{d^m}{dx^m}(e^{xh(t)})=(xh_1(t)+x^2h_1(t)+\cdots+x^mh_m(t))e^{xh(t)}, \tag{18}$$

where every $h_i(t)$ is a product of derivatives of $h(t)$.

Let us denote $r(k,m,x)=\sum_{j=0}^{k}\binom{k}{j}p_{j+m}(x)s_{k-j}(1-x)$. Expanding $\frac{1}{s(h(t))}h_i(t)e^{h(t)}=\sum_{k\geq 0}\alpha_{ik}\frac{t^k}{k!}$, from (17) and (18) we get $r(k,m,x)=x\alpha_{1k}+x^2\alpha_{2k}+\cdots+x^m\alpha_{mk}$.

Because $d_{k,n}(x)=r(n-k,k,x)/s_n(1)$ we obtain $deg(d_{k,n}(x))=k$.

Suppose that $p\in P_m$. Then $\left[0,\frac{1}{n},\ldots,\frac{k}{n};p\right]=0$ for $k\geq m+1$, and using (15) we get $L_n^{Q,S}(P_m)\subseteq P_m$. □

REMARK 1. For $Q = D$ (it means that s_n is an Appell set A_n) we have $\theta = X$, therefore in this case $d_{k,n} = \frac{A_{n-k}(1)}{A_n(1)} x^k$. This representation for operators constructed with Appell sequences was given by C. Manole in [5].

THEOREM 4. *Suppose that all the assumptions of Theorem 2 are true, then there exists θ_{1n}, θ_{2n}, $\theta_{3n} \in [0,1]$ such that $\forall x \in [0,1]$ and $\forall f \in C[0,1]$ we have*

$$(L_n^{Q,S} f)(x) = f(a_n x) + \alpha(x,n)[\theta_{1n},\ \theta_{2n},\ \theta_{3n};\ f]$$

where $\alpha(x,n) = x^2(b_n - a_n^2) + x(a_n - b_n - c_n)$.

Proof. First we shall prove that $f(a_n x) \le (L_n^{Q,S} f)(x)$ for every convex function f.

Let us denote $c_k = \frac{1}{s_n(1)}\binom{n}{k} p_k(x) s_{n-k}(1-x)$ and $x_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$.

We have $c_k \ge 0$, $\sum_{k=0}^n c_k = 1$ and $x_k > 0$, $\forall k \in \mathbb{N}$. If f is a convex function then $f(\sum_{k=0}^n c_k x_k) \le \sum_{k=0}^n c_k f(x_k)$; but

$$\sum_{k=0}^n c_k x_k = (L_n^{Q,S} e_1)(x) = a_n x \quad \text{and} \quad \sum_{k=0}^n c_k f(x_k) = (L_n^{Q,S} f)(x)$$

therefore we get $f(a_n x) \le (L_n^{Q,S} f)(x)$.

If we consider the formula

$$f(a_n x) = (L_n^{Q,S} f)(x) + (R_n f)(x)$$

we have $(R_n f) \le 0$ for every convex function f.

Since $(R_n e_i)(x) = 0$ for $i = 0, 1$, the degree of exactness of the previous formula is one and then there exist $\theta_{1n},\ \theta_{2n},\ \theta_{3n} \in [0,1]$ such that the remainder can be represented in the following form

$$(R_n f)(x) = (R_n e_2)(x)[\theta_{1n},\ \theta_{2n},\ \theta_{3n};\ f]$$

where $(R_n e_2)(x) = x^2(a_n^2 - b_n) + x(b_n + c_n - a_n)$, so we obtain the conclusion. □

4. EXAMPLES

1. If $S = I$ then $s_n = p_n$ and in this case the operator defined by (2) becomes the binomial operator (1) introduced by Tiberiu Popoviciu in [9].

1.1. For $Q = D$ the basic sequence is $p_n(x) = x^n$ and $L_n^{D,I}$ is the Bernstein operator B_n.

1.2. If Q is Abel operator $A = E^{-\beta}D$ we have $p_n(x) = x(x+n\beta)^{n-1}$ and $L_n^{A,I}$ is the second operator introduced by Cheney and Sharma in [1],

$$(L_n^{A,I}f)(x) = \\ = \frac{1}{(1+n\beta)^{n-1}}\sum_{k=0}^{n}\binom{n}{k}x(x+k\beta)^{k-1}(1-x)(1-x+(n-k)\beta)^{n-k-1}f\left(\frac{k}{n}\right)$$

1.3. For Laguerre delta operator $L = \frac{D}{D+I}$ the basic sequence is $l_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(n-1)!}{(k-1)!}x^k$ and the coresponding binomial operator has been considered by T. Popoviciu.

1.4. The delta operator $Q = \frac{1}{\alpha}\nabla_\alpha = \frac{1}{\alpha}(I - E^{-\alpha})$ has the basic sequence $p_n(x) = x^{[n,-\alpha]} = x(x+\alpha)\dots(x+(n-1)\alpha)$ and in this case we obtain the operator

$$(S_nf)(x) = \frac{1}{1^{[n,-\alpha]}}\sum_{k=0}^{n}\binom{n}{k}x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}f(\tfrac{k}{n})$$

which has been introduced and investigated in detail by D. D. Stancu in [14], [16] and other papers.

1.5. The exponential polynomials $t_n(x) = \sum_{k=0}^{n} S(n,k)x^k = e^{-x}\sum_{k=0}^{\infty}\frac{k^n x^k}{k!}$, where $S(n,k)$ denote the Stirling numbers of the second kind, are basic polynomials for the delta operator $T = \ln(I+D)$. The approximation operator construct by means of the exponential polynomials

$$(L_n^Tf)(x) = \frac{1}{t_n(1)}\sum_{k=0}^{n}\binom{n}{k}t_k(x)t_{n-k}(1-x)f(\tfrac{k}{n})$$

was studied by C. Manole in [5].

1.6. If we take the delta operator $Q = G = \frac{1}{\alpha}E^{-\beta}\nabla_\alpha = \frac{1}{\alpha}(E^{-\beta} - E^{-\alpha-\beta})$ its basic sequence is $p_n(x) = x(x+\alpha+n\beta)^{[n-1,-\alpha]}$ and the operator

$$(L_n^Gf)(x) = \frac{1}{(1+n\beta)^{[n,-\alpha]}}\cdot \\ \cdot\sum_{k=0}^{n}\binom{n}{k}x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x)(1-x+(n-k)\beta)^{[n-k,-\alpha]}f\left(\frac{k}{n}\right)$$

was investigated by D. D. Stancu, G. Moldovan. In [18] D. D. Stancu and M. R. Occorsio have studied this operator with the nodes $\frac{k+\gamma}{n+\delta}$, $0 \le \gamma \le \delta$.

2. If $Q = D$ and S is an invertible shift invariant operator then $p_n(x) = x^n$ and $s_n = A_n = S^{-1}x^n$ is an Appell set. The operator of the form

$$(L_n^{D,S} f)(x) = \frac{1}{A_n(1)} \sum_{k=0}^{n} \binom{n}{k} x^k A_{n-k}(1-x) f(\tfrac{k}{n})$$

was introduced and investigated by C. Manole in [5].

2.1. If $S = (I+D)^{-1}$ the coresponding Appell set is $A_n(x) = x^n + nx^{n-1}$ and then

$$(L_n^{D,(I+D)^{-1}} f)(x) = \tfrac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}(n-k+1-x) f(\tfrac{k}{n}).$$

3. If we take $Q = A = E^{-\beta}D$ and $S = E^{\beta}Q' = I - \beta D$ then $p_n(x) = x(x+n\beta)^{n-1}$ is the basic sequence for Q and $s_n(x) = (x+n\beta)^n$ a Sheffer set for Q we obtain the first operator introduced by Cheney and Sharma in [1]:

$$(L_n^{A,I-\beta D} f)(x) = \tfrac{1}{(1+n\beta)^n} \sum_{k=0}^{n} \binom{n}{k} x(x+k\beta)^{k-1}(1-x+(n-k)\beta)^{n-k} f\left(\tfrac{k}{n}\right)$$

4. For $Q = \frac{1}{\alpha}E^{-\beta}\nabla_\alpha = \frac{1}{\alpha}(E^{-\beta} - E^{-\alpha-\beta})$ and $S = E^{\alpha+\beta}Q' = \frac{1}{\alpha}((\alpha+\beta)I - \beta E^{\alpha})$ we have $p_n(x) = x(x+\alpha+n\beta)^{[n-1,-\alpha]}$ and $s_n(x) = (x+n\beta)^{[n,-\alpha]}$ therefore the operator $L_n^{Q,S}$ in this case is

$$(L_n^{[\alpha,\beta]} f)(x) =$$
$$= \tfrac{1}{(1+n\beta)^{[n,-\alpha]}} \sum_{k=0}^{n} \binom{n}{k} x(x+\alpha+k\beta)^{[k-1,-\alpha]}(1-x+(n-k)\beta)^{[n-k,-\alpha]}\ f\left(\tfrac{k}{n}\right)$$

If we replace x with $s(x)$ we obtain a operator which has been studied by G. Moldovan in [6]. He has found the value of this operator for the monomials e_i for $i = 1,2$ using some generalized identities of Vandermonde type.

We want to find the sequences $a_n,\ b_n,\ c_n$, which appears in $(L_n^{[\alpha,\beta]} e_i)(x)$, using relations (9).

The Pincherle derivative of Q is

$$Q' = -\frac{\beta}{\alpha}E^{-\beta} + \left(1+\frac{\beta}{\alpha}\right)E^{-\alpha-\beta} = E^{-\alpha-\beta}\left(I - \frac{\beta}{\alpha}\Delta_\alpha\right)$$

so

$$(Q')^{-1} = E^{\alpha+\beta}\sum_{k\geq 0}\beta^k\left(\frac{\Delta_\alpha}{\alpha}\right)^k.$$

Since $s_{n-1}(x) = (x+(n-1)\beta)^{[n-1,-\alpha]} = E^{(n-2)\alpha+(n-1)\beta}x^{[n-1,\alpha]}$ and $x^{[n,\alpha]} = x(x-1)\dots(x-(n-1)\alpha)$ is the basic sequence for the delta operator $\frac{\Delta_\alpha}{\alpha}$, we have $\left(\frac{\Delta_\alpha}{\alpha}\right)^k s_{n-1}(x) = E^{(n-2)\alpha+(n-1)\beta}\left(\frac{\Delta_\alpha}{\alpha}\right)^k x^{[n-1,\alpha]} = (n-1)^{[k]}E^{(n-2)\alpha+(n-1)\beta}\cdot$ $\cdot x^{[n-1-k,\alpha]}$. Because $a_n = \frac{[(Q')^{-1}s_{n-1}](1)}{s_n(1)}$ we get

$$a_n = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{k!\beta^k}{(1+n\beta)^{[k+1,-\alpha]}}.$$

Since $b_n = \frac{n-1}{n}\frac{[(Q')^{-2}s_{n-2}](1)}{s_n(1)}$ and

$$(Q')^{-2} = E^{2\alpha+2\beta}\left(I-\frac{\beta}{\alpha}\Delta_\alpha\right)^{-2} = E^{2\alpha+2\beta}\sum_{k\geq 0}(k+1)\,\beta^k\left(\frac{\Delta_\alpha}{\alpha}\right)^k$$

we obtain

$$b_n = \frac{n-1}{n}\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{(k+1)!\beta^k}{(1+n\beta)^{[k+2,-\alpha]}}.$$

The Pincherle derivative of S^{-1} may be written in the form

$$\begin{aligned}\left(S^{-1}\right)' &= -\,(\alpha+\beta)E^{-\alpha-\beta}(Q')^{-1} - E^{-\alpha-\beta}(Q')^{-2}Q''\\ &= -\,E^{-\alpha-\beta}(Q')^{-2}((\alpha+\beta)Q' + Q'').\end{aligned}$$

Because $Q'' = \frac{1}{\alpha}(\beta^2E^{-\beta} - (\alpha+\beta)^2E^{-\alpha-\beta})$ and $(\alpha+\beta)Q' + Q'' = -\beta E^{-\beta}$ this implies

$$\begin{aligned}(Q')^{-2}(S^{-1})'S &= \beta(Q')^{-3}E^{-\beta}\\ &= \beta E^{3\alpha+2\beta}(I - \frac{\beta}{\alpha}\Delta_\alpha)^{-3}\\ &= E^{3\alpha+2\beta}\sum_{k\geq 0}\frac{(k+1)(k+2)}{2}\beta^{k+1}\left(\frac{1}{\alpha}\Delta_\alpha\right)^k.\end{aligned}$$

From (9) and the previous relation we get

$$c_n = \frac{n-1}{2n}\left((1+n\alpha+n\beta)\sum_{k=0}^{n-3}\binom{n-2}{k}\frac{(k+2)!\beta^{k+1}}{(1+n\beta)^{[k+3,-\alpha]}} + \frac{n!\beta^{n-1}}{(1+n\beta)^{[n,-\alpha]}}\right).$$

5. EVALUATION OF THE ORDERS OF APPROXIMATION

Now we establish some estimates of the order of approximation of a function $f \in C[0,1]$ by means of the operator $L_n^{Q,S}$, defined by (2).

According to a result of O. Shisha and B. Mond [13], we can write

$$\left|f(x) - \left(L_n^{Q,S}f\right)(x)\right| \le \left[1 + \tfrac{1}{\delta^2}L_n^{Q,S}\left((t-x)^2;x\right)\right]\omega_1(f;\delta), \quad \delta \in \mathbb{R}^+$$

Using the relations (8) we have

$$L_n^{Q,S}\left((t-x)^2;x\right) = x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)$$

so we get

$$\left|f(x) - \left(L_n^{Q,S}f\right)(x)\right| \le \left[1 + \tfrac{1}{\delta^2}\left[x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)\right]\right]\omega_1(f;\delta).$$

One observes that if $b_n - 2a_n + 1 < 0$ then $x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n) \le \frac{(a_n-b_n-c_n)^2}{4(2a_n-b_n-1)}$, $\forall x \in [0,1]$.

By choosing $\delta = \frac{1}{\sqrt{n}}$ we can state

THEOREM 5. *If $f \in C[0,1]$ and $\exists k \in \mathbb{N}$ such as $b_n - 2a_n + 1 < 0$, $\forall n \ge k$, then we can give the following estimation of the order of approximation, by means of the first modulus of continuity*

$$\left\|f - L_n^{Q,S}f\right\| \le \left(1 + \tfrac{n}{4}\tfrac{(a_n-b_n-c_n)^2}{(2a_n-b_n-1)}\right)\omega_1\left(f;\tfrac{1}{\sqrt{n}}\right), \quad n \ge k,$$

where a_n, b_n, c_n are defined by (9).

In the case of binomial operators of positive type defined by (1), since $S' = I' = O$ we have

$$a_n = 1, \quad c_n = 0, \quad b_n = \frac{n-1}{n}\frac{[(Q')^{-2}p_{n-2}](1)}{p_n(1)}. \tag{19}$$

Then $b_n - 2a_n + 1 = b_n - 1 < 0$, $\forall n \in \mathbb{N}$ therefore the previous inequality reduces to

$$\|f - L_n f\| \le \left(\frac{5}{4} + \frac{n}{4} d_n\right) \omega_1\left(f; \frac{1}{\sqrt{n}}\right),$$

where

$$d_n = \frac{n-1}{n} - b_n = \frac{n-1}{n}\left(1 - \frac{[(Q')^{-2} p_{n-2}](1)}{p_n(1)}\right) \tag{20}$$

We mention that this inequality was established by D. D. Stancu in [18].

In order to find an evaluation of the order of approximation using both moduli of smoothness ω_1 and ω_2 we can use a result of H. H. Gonska and R. K. Kovacheva included in the following

Lemma 3. [2]. *If $I = [a, b]$ is a compact interval of the real axis and $I_1 = [a_1, b_1]$ is a subinterval of it, and if we assume that $L : C(I) \to C(I_1)$ is a positive operator, such that $Le_0 = e_0$ and $0 \le \delta \le \frac{1}{2}(b - a)$, then we have*

$$\begin{aligned} |f(x) - L(f(t); x)| \le \; & \tfrac{2}{\delta} |L(t - x; x)| \, \omega_1(f; \delta) + \\ & + \tfrac{3}{2}\left[1 + \tfrac{1}{\delta}|L(t - x; x)| + \tfrac{1}{2\delta^2} L((t - x)^2; x)\right] \omega_2(f; \delta). \end{aligned}$$

Using the relations (8) we obtain the inequality

$$\begin{aligned} & \left| f(x) - \left(L_n^{Q,S} f\right)(x) \right| \le \tfrac{2}{\delta} |(a_n - 1)x| \, \omega_1(f; \delta) + \\ & + \tfrac{3}{2}\left[1 + \tfrac{1}{\delta}|(a_n - 1)x| + \tfrac{1}{2\delta^2}\left[x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)\right]\right] \omega_2(f; \delta). \end{aligned}$$

If $b_n - 2a_n + 1 < 0$ the previous inequality implies

$$\left\| f - L_n^{Q,S} f \right\| \le \tfrac{2}{\delta}(1 - a_n)\omega_1(f; \delta) + \tfrac{3}{2}\left[1 + \tfrac{1}{\delta}(1 - a_n) + \tfrac{1}{8\delta^2} \tfrac{(a_n - b_n - c_n)^2}{(2a_n - b_n - 1)}\right] \omega_2(f; \delta).$$

By choosing $\delta = \frac{1}{\sqrt{n}}$ we get

$$\begin{aligned} \left\| f - L_n^{Q,S} f \right\| \le \; & 2\sqrt{n}(1 - a_n)\omega_1(f; \tfrac{1}{\sqrt{n}}) + \\ & + \tfrac{3}{2}\left[1 + \sqrt{n}(1 - a_n) + \tfrac{n}{8} \tfrac{(a_n - b_n - c_n)^2}{(2a_n - b_n - 1)}\right] \omega_2(f; \tfrac{1}{\sqrt{n}}). \end{aligned}$$

If we consider the binomial operator introduced by Tiberiu Popoviciu, using (19) and the previous relation, we arrive at an inequality which has found by D. D. Stancu (see [18])

$$\|f - L_n f\| \leq \tfrac{3}{16}\,(9 + nd_n)\,\omega_2(f; \tfrac{1}{\sqrt{n}}),$$

where d_n is defined by (20).

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