HARMONIC BLENDING APPROXIMATION

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Dedicated to Professor Werner Haussmann on his sixties birthday

Abstract. The concept of harmonic Hilbert space $H_D(\mathbb{R}^n)$ was introduced in [2] as an extension of periodic Hilbert spaces [1], [2], [5], [6]. In [4] we introduced multivariate harmonic Hilbert spaces and studied approximation by exponential-type function in these spaces and derived error bounds in the uniform norm for special functions of exponential type which are defined by Fourier partial integrals $S_b(f)$:

$$S_b(f)(x) = \int_{\mathbb{R}^n} \chi_{[-b,b]}(t)F(t)\exp(i(t,x))dt,$$

$[-b,b] = [-b_1,b_1] \times \ldots \times [-b_n,b_n]$, $b_1 > 0, \ldots, b_n > 0$, where

$$F(t) \sim \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} f(x)\exp(-i(x,t))dx \in L_2(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$$

is the Fourier transform of $f \in L_2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. In this paper we will investigate more general approximation operators $S_\psi$ in harmonic Hilbert spaces of tensor product type.


1. HARMONIC HILBERT SPACES

The function $D$ is called the defining function of the harmonic Hilbert space $H_D(\mathbb{R}^n)$. It satisfies the following conditions:

$$D(-t) = D(t), \quad 0 \leq D(t) \leq 1, \quad D \in L_1(\mathbb{R}^n) \quad (\Rightarrow D \in L_2(\mathbb{R}^n)).$$

The Fourier integral of the defining function is called the generating function of the harmonic Hilbert space:

$$d(x) = \int_{\mathbb{R}^n} D(t)\exp(i(x,t))dt \in L_2(\mathbb{R}^n).$$

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The generating function is a function from the Wiener algebra $A(\mathbb{R}^n)$. This algebra is defined as the set of functions

$$f(x) = \int_{\mathbb{R}^n} F(t) \exp(i(x,t))dt,$$

$$F(t) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^n} f(x) \exp(-i(x,t))dx \in L_1(\mathbb{R}^n).$$

It is a subalgebra of the algebra of uniformly continuous functions on the real line vanishing at infinity $C_0(\mathbb{R}^n)$. The norm of this algebra is the maximum norm:

$$\|f\|_\infty = \sup\{ |f(x)| : x \in \mathbb{R}^n \}.$$

The norm of the Wiener algebra is given by

$$\|f\|_a = \int_{\mathbb{R}^n} |F(t)| dt.$$

The inequality $\|f\|_\infty \leq \|f\|_a$ holds for any function of the Wiener algebra. Note that $F \geq 0$ implies $\|f\|_\infty = \|f\|_a$.

The inner product of the harmonic Hilbert space is defined by

$$(f,g)_D = \int_{\mathbb{R}^n} F(t)\overline{G(t)} \frac{1}{D(t)} dt.$$

It is a reproducing kernel Hilbert space:

$$f(x) = (f,d(\cdot - x))_D.$$

Any harmonic Hilbert space is a subspace of the Wiener algebra:

$$H_D(\mathbb{R}^n) \subseteq A(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n).$$

The imbeddings are continuous due to the estimates

$$\|f\|_\infty \leq \|f\|_a \leq \sqrt{d(0)} \|f\|_D.$$

Examples of defining functions in the univariate case are taken from summability theory. We give a list of typical examples:

- **Sobolev space** $W^1(\mathbb{R})$:
  
  $$D(t) = \frac{1}{1 + t^2}, \quad d(x) = \pi \exp(-|x|),$$

- **Holomorphic Sobolev space** $H^1(\mathbb{R})$:
  
  $$D(t) = \exp(-|t|), \quad d(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2},$$

- **Paley–Wiener space** $PW_0(\mathbb{R})$:
  
  $$D(t) = (b - |t|)_+, \quad d(x) = 2 \sin(bx)/x.$$
Tensor product harmonic Hilbert spaces are obtained by choosing tensor products of univariate defining functions. For notational simplicity we consider mainly the case $n = 2$:

$$
D^P(t_1, t_2) = D_1(t_1)D_2(t_2) = (D_1 \otimes D_2)(t_1, t_2),
$$

$$
d^P(x_1, x_2) = d_1(x_1)d_2(x_2) = (d_1 \otimes d_2)(x_1, x_2),
$$

$$
H_{D^p}(\mathbb{R}^2) = H_{D_1}(\mathbb{R}) \otimes H_{D_2}(\mathbb{R}).
$$

In our examples from the univariate case we use the following notations.

Tensor product Sobolev space $W^{(1,1)}(\mathbb{R}^2)$:

$$
D(t_1, t_2) = \frac{1}{1+t_1^2} \cdot \frac{1}{1+t_2^2}, \quad d(x_1, x_2) = \pi^2 \exp(-|x_1|) \exp(-|x_2|).
$$

Tensor product holomorphic Sobolev space $H^{(1,1)}(\mathbb{R}^2)$:

$$
D(t_1, t_2) = \exp(-|t_1|) \exp(-|t_2|), \quad d(x_1, x_2) = \frac{1}{\pi^2} \cdot \frac{1}{1+x_1^2} \cdot \frac{1}{1+x_2^2}.
$$

Tensor product holomorphic Paley–Wiener space $PW_{b_1,b_2}^{(1,1)}(\mathbb{R}^2)$:

$$
D(t_1, t_2) = (b_1 - |t_1|)_+^0 (b_2 - |t_2|)_+^0, \quad d(x_1, x_2) = 4 \frac{\sin(b_1 x_1)}{x_1} \cdot \frac{\sin(b_2 x_2)}{x_2}.
$$

## 2. $\psi$-FOURIER PARTIAL INTEGRALS

We denote the set of functions $\psi \in L_\infty(\mathbb{R}^n)$ satisfying $0 \leq \psi(t) \leq 1$, $t \in \mathbb{R}^n$, by $L_\infty(\mathbb{R}^n, [0, 1])$.

The function $\psi \in L_\infty(\mathbb{R}^n, [0, 1])$ is used to define the $\psi$-Fourier partial integral

$$
S_\psi(f)(x) = \int_{\mathbb{R}^n} \psi(t) F(t) \exp(i(x, t)) \, dt
$$

as an approximation of the Fourier integral

$$
f(x) = S_1(f)(x) = \int_{\mathbb{R}^n} F(t) \exp(i(x, t)) \, dt.
$$

The classical Fourier partial integral with respect to the interval $[-b, b]$ is given by the characteristic function
\[ \psi(t) = \chi_{[-b,b]}(t), \]

\[ S_\psi(f)(x) = \int_{\mathbb{R}^n} \chi_{[-b,b]}(t)F(t) \exp(i(x,t))dt =: S_b(f)(x). \]

In the univariate case we have

\[ \psi(t) = \left(1 - \frac{|t|}{b}\right)_+^0, \]

\[ S_\psi(f)(x) = \int_{-b}^{b} F(t) \exp(ixt)dt =: S_b(f)(x). \]

The Fejér-partial integral is given by

\[ \phi(t) = \left(1 - \frac{|t|}{b}\right)_-^1, \]

\[ S_\phi(f)(x) = \int_{-b}^{b} \left(1 - \frac{|t|}{b}\right)_-^1 F(t) \exp(ixt)dt =: F_b(f)(x). \]

\( S_\psi \) is a bounded linear operator on \( A(\mathbb{R}^n) \). It satisfies the norm inequality

\[ \|S_\psi(f)\|_a \leq \|f\|_a. \]

The restriction of \( S_\psi \) to the harmonic Hilbert space \( H_D(\mathbb{R}^n) \) defines a bounded linear self-adjoint operator:

\[ (S_\psi(f),g)_D = (S_\psi(f),g)_D. \]

Moreover, the following estimate holds

\[ \|S_\psi(f)\|_D \leq \|f\|_D. \]

\( S_\psi \) is a projector if and only if

\[ \psi^2 = \psi \iff S_\psi^2 = S_\psi. \]

In this case \( S_\psi \) is a projector on \( A(\mathbb{R}^n) \) and induces by restriction an orthogonal projector on \( H_D(\mathbb{R}^n) \).

The approximation order of the \( \psi \)-Fourier integral in the harmonic Hilbert space \( H_D(\mathbb{R}^n) \) is determined by the remainder of the generating function \( d - S_\psi(d) \).
Proposition 2.1. Assume that $f \in H_D(\mathbb{R}^n)$. Then the error estimate
\[
\|f - S_\psi(f)\|_\infty \leq \|f\|_D \|1 - \psi\|_{2,D}
\]
holds with
\[
\|d - S_\psi(d)\|_D = \sqrt{\int_{\mathbb{R}^n} (1 - \psi(t))^2 D(t) dt} = \|1 - \psi\|_{2,D}.
\]

Proof. The structure of the harmonic Hilbert space as a reproducing kernel Hilbert space implies
\[
(f, d(\cdot - x))_D = f(x)
\]
in view of
\[
d(y - x) = \int_{\mathbb{R}^n} \exp(-i(x,t)) D(t) \exp(i(y,t)) dt
\]
and
\[
f(x) = \int_{\mathbb{R}^n} F(t) \overline{D(t) \exp(-i(x,t))} D(t) dt = (f, d(\cdot - x))_D.
\]
Moreover, we have
\[
S_\phi(f)(x) = \int_{\mathbb{R}^n} \phi(t) F(t) \exp(i(x,t)) dt
\]
which implies
\[
\|S_\phi(f)\|_D^2 = \int_{\mathbb{R}^n} \phi(t)^2 |F(t)|^2 / D(t) dt.
\]
The translation operator and the $\psi$-Fourier partial integral operator commute:
\[
S_\psi(d(\cdot - x))(y) = \int_{\mathbb{R}^n} \psi(t) \exp(-i(x,t)) D(t) \exp(i(y,t)) dt = S_\psi(d)(y - x),
\]
i.e., we have
\[
S_\psi(d(\cdot - x)) = S_\psi(d(\cdot - x)).
\]
Next we can conclude
\[
f(x) - S_\psi(f)(x) = (f, d(\cdot - x))_D - (S_\psi(f), d(\cdot - x))_D = (f, d(\cdot - x))_D - (f, S_\psi(d(\cdot - x)))_D,
\]
i.e., we have
\[
f(x) - S_\psi(f)(x) = (f, d(\cdot - x) - S_\psi(d(\cdot - x)))_D = (f, S_{1-\psi}(d(\cdot - x)))_D.
\]
Consider the linear functional on $H_D(\mathbb{R}^n)$ defined by
\[
L_{\psi,x}(f) = f(x) - S_\psi(f)(x) = (f, S_{1-\psi}(d(\cdot - x)))_D.
\]
By the Riesz representation theorem in Hilbert spaces [A.N. Michel, C.J. Herget: Applied Linear Algebra and Functional Analysis] its norm is given by
\[ \|L_{\psi,x}\|_D = \|S_1 - \psi(d(\cdot - x))\|_D = \|S_1 - \psi(d)\|_D = \|d - S_\psi(d)\|_D. \]

Since
\[ \|d - S_\psi(d)\|_D^2 = \int_{\mathbb{R}^n} (1 - \psi(t))^2 D(t) dt \]
the proof is complete. \(\square\)

**Remark 2.1.** If \(S_\psi\) is an orthogonal projector the sharper estimate
\[ \|f - S_\psi(f)\|_\infty \leq \|f - S_\psi(f)\|_D \|d - S_\psi(d)\|_D \]
holds.

This follows from
\[ f(x) - S_\psi(f)(x) = (f, S_1 - \psi(d(\cdot - x)))_D = (S_1 - \psi(f), S_1 - \psi(d(\cdot - x)))_D \]
by an application of the Cauchy–Schwarz inequality.

### 3. LATTICES OF FOURIER PARTIAL INTEGRAL OPERATORS

We denote the set of functions \(\psi \in L_\infty(\mathbb{R}^n)\) satisfying \(0 \leq \psi(t) \leq 1\), \(t \in \mathbb{R}^n\), by \(L_\infty(\mathbb{R}^n, [0, 1])\). We summarize some algebraic properties of \(L_\infty(\mathbb{R}^n, [0, 1])\):

1. \(\psi \in L_\infty(\mathbb{R}^n, [0, 1]) \Rightarrow 1 - \psi \in L_\infty(\mathbb{R}^n, [0, 1]).\)
2. \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1]) \Rightarrow \psi \cdot \gamma \in L_\infty(\mathbb{R}^n, [0, 1]).\)
3. \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1]) \Rightarrow \psi \oplus \gamma := \psi + \gamma - \psi \cdot \gamma \in L_\infty(\mathbb{R}^n, [0, 1]).\)
4. \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1]) \Rightarrow \psi \lor \gamma := \max\{\psi, \gamma\} \in L_\infty(\mathbb{R}^n, [0, 1]).\)
5. \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1]) \Rightarrow \psi \land \gamma := \min\{\psi, \gamma\} \in L_\infty(\mathbb{R}^n, [0, 1]).\)
6. \(\psi \cdot \gamma \leq \min\{\psi, \gamma\} \leq \max\{\psi, \gamma\} \leq \psi \oplus \gamma.\)

This shows that \(L_\infty(\mathbb{R}^n, [0, 1])\) is a lattice of real valued measurable functions. The following result is easily verified.

**Proposition 3.1.** Assume \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1])\). Then we have
\[ 1 - \psi \cdot \gamma = (1 - \psi) \oplus (1 - \gamma) = (1 - \psi) + (1 - \gamma) - (1 - \psi) \cdot (1 - \gamma), \]
\[ 1 - \psi \oplus \gamma = (1 - \psi) \cdot (1 - \gamma). \]

The set of commuting non negative Hermitian operators \(S_\psi\) forms an operator lattice with respect to the order relation:
\[
S_\psi \geq 0 \iff (S_\psi f, f)_D \geq 0, \forall f \in H_D(\mathbb{R}^n).
\]

Consider any two functions \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1])\). The associated Fourier partial integral operators \(S_\psi, S_\gamma\) commute and their product is again a \textit{product Fourier partial integral operator} satisfying

\[
S_\psi S_\gamma = S_\gamma S_\psi = S_{\psi\gamma}.
\]

The Boolean sum of \(\psi, \gamma \in L_\infty(\mathbb{R}^n, [0, 1])\) defines the \textit{blending Fourier partial integral operator}:

\[
S_\psi \oplus S_\gamma = S_\gamma + S_\psi - S_{\psi\gamma}.
\]

It is important to note that

\[
\psi^2 = \psi \Rightarrow (1 - \psi)^2 = 1 - \psi
\]

\[
\psi^2 = \psi, \gamma^2 = \gamma \Rightarrow (\psi\gamma)^2 = \psi\gamma, (\psi \oplus \gamma)^2 = \psi \oplus \gamma.
\]

Note that the characteristic functions \(\chi_M, \chi_N\) satisfy the above conditions. In particular we have

\[
\chi_M \cdot \chi_N = \chi_{M \cap N}, \quad \chi_M \oplus \chi_N = \chi_{M \cup N}, \quad 1 - \chi_M = \chi_{M^c},
\]

with \(M^c = \mathbb{R}^n - M\).

**Proposition 3.2.** The set of operators \(S_\psi\) with \(\psi^2 = \psi\) form a Boolean algebra of commuting projectors

\[
B := \{S_\psi : \psi \in L_\infty(\mathbb{R}^n, [0, 1]), \psi^2 = \psi\}.
\]

This aspect turns out to be useful in the multivariate setting.

We first determine the approximation order of product approximation \(S_\psi S_\gamma = S_{\psi\gamma}\).

**Proposition 3.3.** Assume that \(f \in H_D(\mathbb{R}^n)\). Then the error estimate

\[
\|f - S_\psi S_\gamma(f)\|_\infty \leq \|f\|_D \left(\|1 - \psi\|_{2,D} + \|1 - \gamma\|_{2,D}\right)
\]

holds.

**Proof.** By Proposition 2.1 we have

\[
\|f - S_\psi S_\gamma(f)\|_\infty \leq \|f\|_D \|d - S_\psi S_\gamma(d)\|_D.
\]
Since
\[ \|d - S_\psi S_\gamma(d)\|_D = \|d - S_\psi(d) + S_\gamma(d - S_\psi(d))\|_D \]
we obtain
\[ \|d - S_\psi S_\gamma(d)\|_D \leq \|d - S_\psi(d)\|_D + \|S_\gamma(d - S_\psi(d))\|_D \]
\[ \leq \|d - S_\psi(d)\|_D + \|d - S_\psi(d)\|_D \]
\[ = \|1 - \psi\|_{2,D} + \|1 - \gamma\|_{2,D}. \]
\[ \square \]

Next we determine the approximation order of blending approximation \[ S_\psi + S_\gamma - S_\psi \cdot S_\gamma = S_\psi \oplus S_\gamma. \]

**Proposition 3.4.** Assume that \( f \in H_D(\mathbb{R}^n) \). Then the error estimate
\[ \|f - (S_\psi \oplus S_\gamma)(f)\|_\infty \leq \|f\|_D \sqrt{\|(1 - \psi)^2\|_{2,D} \sqrt{\|(1 - \gamma)^2\|_{2,D}}} \]
holds.

**Proof.** By Proposition 2.1 we have
\[ \|f - S_\psi \otimes S_\gamma(f)\|_\infty \leq \|f\|_D \|d - S_\psi \otimes S_\gamma(d)\|_D \]
Since
\[ \|S_1 - \gamma \oplus \psi(d)\|_D^2 = \int_{\mathbb{R}^n} (1 - \gamma(t) \otimes \psi(t))^2 D(t) dt \]
\[ = \int_{\mathbb{R}^n} (1 - \gamma(t))^2 (1 - \psi(t))^2 D(t) dt \]
\[ \leq \sqrt{\int_{\mathbb{R}^n} (1 - \gamma(t))^4 D(t) dt} \sqrt{\int_{\mathbb{R}^n} (1 - \psi(t))^4 D(t) dt} \]
\[ = \left\| (1 - \psi)^2 \right\|_{2,D} \cdot \left\| (1 - \gamma)^2 \right\|_{2,D} \]
the proof is complete. \[ \square \]

As a special case we obtain

**Proposition 3.5.** Assume that \( f \in H_D(\mathbb{R}^n) \). Then the error estimate
\[ \|f - (S_\psi \oplus S_\psi)(f)\|_\infty \leq \|f\|_D \|1 - \psi\|^2_{2,D} \]
holds.

4. APPROXIMATION IN TENSOR PRODUCT HARMONIC HILBERT SPACES

In the tensor product harmonic Hilbert space
\[ H_D^p(\mathbb{R}^2) = H_D(\mathbb{R}) \otimes H_D(\mathbb{R}) \]
we have the simple situation
\[ D^P(t_1, t_2) = D(t_1)D(t_2) = D \otimes D(t_1, t_2), \]
\[ d^P(x_1, x_2) = d(x_1)d(x_2) = d \otimes d(x_1, x_2). \]

This leads to special constructions of \( \psi \)-Fourier integrals choosing tensor products of measurable functions:
\[ \gamma(t_1, t_2) = \phi(t_1) = \phi \otimes 1_\mathbb{R}(t_1, t_2), \]
\[ \psi(t_1, t_2) = \zeta(t_2) = 1_\mathbb{R} \otimes \zeta(t_1, t_2). \]

Recall that
\[ \|d - S_\psi(d)\|_D = \sqrt{\int_{\mathbb{R}^n} (1 - \psi(t))^2 D(t)dt} = \|1 - \psi\|_{2,D}. \]
The tensor product structure implies
\[ \left\| d^P - S_{\phi \otimes 1_\mathbb{R}}(d^P) \right\|_{D^P} = \left\| S_{(1_\mathbb{R} - \phi) \otimes 1_\mathbb{R}}(d^P) \right\|_{D^P} = \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1 - \phi(t_1))^2 dt_1 dt_2} = \|1 - \phi\|_{2,D} \|1\|_{2,D}, \]
\[ \left\| d^P - S_{1_\mathbb{R} \otimes \zeta}(d^P) \right\|_{D^P} = \left\| S_{1_\mathbb{R} \otimes (1_\mathbb{R} - \zeta)}(d^P) \right\|_{D^P} = \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1 - \zeta(t_2))^2 dt_1 dt_2} = \|1 - \zeta\|_{2,D} \|1\|_{2,D}. \]

An application of Proposition 3.3 yields

**Proposition 4.1.** Assume that \( f \in H_{D}(\mathbb{R}) \otimes H_D(\mathbb{R}) \). Then the error estimate
\[ \|f - S_{\phi \otimes \zeta}(f)\|_{\infty} \leq \|f\|_{D^P} \|1\|_{2,D} \left(\|1 - \phi\|_{2,D} + \|1 - \zeta\|_{2,D}\right) \]
holds.

For blending approximation in \( H_{D^P}(\mathbb{R}^2) \) with the operator
\[ S_{\phi \otimes 1_\mathbb{R}} \oplus S_{1_\mathbb{R} \otimes \zeta} = S_{1_\mathbb{R} \otimes \zeta} + S_{\phi \otimes 1_\mathbb{R}} - S_{\phi \otimes \zeta} \]
it follows
\[ \left\| d^P - S_{\phi \otimes 1_\mathbb{R}} \oplus S_{1_\mathbb{R} \otimes \zeta}(d^P) \right\|_{D^P} = \left\| S_{(1_\mathbb{R} - \phi) \otimes (1_\mathbb{R} - \zeta)}(d^P) \right\|_{D^P} = \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1 - \phi(t_1))^2(1 - \zeta(t_2))^2 dt_1 dt_2} = \|1 - \phi\|_{2,D} \|1 - \zeta\|_{2,D}. \]
Thus we have shown

**Proposition 4.2.** Assume that \( f \in H_D(\mathbb{R}) \otimes H_D(\mathbb{R}) \). Then the error estimate

\[
\| f - S_{\phi \otimes 1_R} \oplus S_{1_R \otimes \zeta}(f) \|_\infty \leq \| f \|_{D^P} \| 1 - \phi \|_{2,D} \| 1 - \zeta \|_{2,D}
\]

holds.

5. EXPONENTIAL-TYPE BLENDING APPROXIMATION

As a classical example let

\[
\gamma(t_1, t_2) = \chi_{[-b_1, b_1]}(t_1), \quad \psi(t_1, t_2) = \chi_{[-b_2, b_2]}(t_2),
\]

In this case the operators are parametrically extended univariate Fourier partial integrals

\[
S_{b_1}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_1, b_1]}(t_1) F(t) \exp(i(x, t)) dt,
\]

\[
S_{b_2}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_2, b_2]}(t_2) F(t) \exp(i(x, t)) dt
\]

These are functions of exponential-type in \( x_1 \), respectively in \( x_2 \). The corresponding product operator \( S_{b_1} S_{b_2} \) is the bivariate Fourier partial integral

\[
S_{b_2} S_{b_1}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_1, b_1]}(t_1) \chi_{[-b_2, b_2]}(t_2) F(t) \exp(i(x, t)) dt = S_{(b_1, b_2)}(f)(x).
\]

These are bivariate functions of exponential-type. Recall that

\[
\| d - S_{\phi}(d) \|_D = \sqrt{\int_{\mathbb{R}^n} (1 - \phi(t))^2 D(t) dt} = \| 1 - \phi \|_{2,D}.
\]

**Proposition 5.1.** The asymptotic error estimate for the bivariate Fourier partial integral follows from the general result Proposition 4.1

\[
\| f - S_{(b_1, b_2)}(f) \|_\infty = \mathcal{O} \left( \sqrt{\int_{b_1}^\infty D(t) dt} + \sqrt{\int_{b_2}^\infty D(t) dt} \right), \quad b_1, b_2 \uparrow \infty.
\]

The Boolean sum \( S_{b_1} \oplus S_{b_2} \) is called the bivariate hyperbolic cross Fourier integral

\[
S_{b_2} \oplus S_{b_1}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_1, b_1]}(t_1) F(t) \exp(i(x, t)) dt + \int_{\mathbb{R}^2} \chi_{[-b_2, b_2]}(t_2) F(t) \exp(i(x, t)) dt
\]

\[
- \int_{\mathbb{R}^2} \chi_{[-b_1, b_1]}(t_1) \chi_{[-b_2, b_2]}(t_2) F(t) \exp(i(x, t)) dt = S_{(b_1, b_2)}(f)(x).
\]

These are bivariate blending functions of exponential-type.
Proposition 5.2. The asymptotic error estimate for the bivariate hyperbolic cross Fourier integral follows from the general result Proposition 4.2

\[ \| f - S^{(b_1, b_2)}(f) \|_\infty = O \left( \int_{b_1}^{\infty} D(t) dt \cdot \int_{b_2}^{\infty} D(t) dt \right), \quad b_1 \vee b_2 \uparrow \infty. \]

The first example is the tensor product Sobolev space. The univariate defining function is

\[ D(t) = \frac{1}{1 + t^2} \]

and it defines the univariate Sobolev space

\[ H_D(\mathbb{R}) = W^1(\mathbb{R}), \]

The bivariate tensor product harmonic Hilbert space is obtained by choosing the tensor product defining function,

\[ H_D^P(\mathbb{R}^2) = W^{1,1}(\mathbb{R}^2). \]

The univariate error norms satisfy the asymptotic relations

\[ \sqrt{\int_{b_1}^{\infty} \frac{1}{1+t^2} dt} = O \left( b_1^{-\frac{1}{2}} \right), \quad \sqrt{\int_{b_2}^{\infty} \frac{1}{1+t^2} dt} = O \left( b_2^{-\frac{1}{2}} \right). \]

Corollary 5.1. The asymptotic error estimate for the bivariate product Fourier partial integral in \( W^{1,1}(\mathbb{R}^2) \) is given by

\[ \| f - S_{(b_1, b_2)}(f) \|_\infty = O \left( b_1^{-\frac{1}{2}} + b_2^{-\frac{1}{2}} \right), \quad b_1 \wedge b_2 \uparrow \infty \]

It is determined by the maximal univariate error bound.

Corollary 5.2. The asymptotic error estimate for the bivariate hyperbolic cross Fourier partial integral in \( W^{1,1}(\mathbb{R}^2) \) is given by

\[ \| f - S^{(b_1, b_2)}(f) \|_\infty = O \left( (b_1 b_2)^{-\frac{1}{2}} \right), \quad b_1 \vee b_2 \uparrow \infty. \]

It is determined by the product of the univariate error bounds.

For the univariate defining function

\[ D(t) = \exp(-|t|) \]

the bivariate tensor product holomorphic Sobolev space is obtained,

\[ H_D^P(\mathbb{R}^2) = H^{1,1}(\mathbb{R}^2). \]
The univariate error norms satisfy the asymptotic relations
\[
\sqrt{\int_{b_1}^{\infty} \exp(-|t|) dt} = \mathcal{O}\left(\exp\left(-\frac{b_1}{2}\right)\right), \quad \sqrt{\int_{b_2}^{\infty} \exp(-|t|) dt} = \mathcal{O}\left(\exp\left(-\frac{b_2}{2}\right)\right).
\]

**Corollary 5.3.** The asymptotic error estimate for the bivariate product Fourier partial integral in \(H^{(1,1)}(\mathbb{R}^2)\) is given by
\[
\|f - S_{(b_1, b_2)}(f)\|_\infty = \mathcal{O}\left(\exp\left(-\frac{b_1}{2}\right) + \exp\left(-\frac{b_2}{2}\right)\right), \quad b_1 \wedge b_2 \uparrow \infty.
\]

It is determined by the maximal univariate error bound.

**Corollary 5.4.** The asymptotic error estimate for the bivariate hyperbolic cross Fourier partial integral in \(H^{(1,1)}(\mathbb{R}^2)\) is given by
\[
\|f - S^{(b_1, b_2)}(f)\|_\infty = \mathcal{O}\left(\exp\left(-\frac{b_1 + b_2}{2}\right)\right), \quad b_1 \vee b_2 \uparrow \infty.
\]

It is determined by the product of the univariate error bounds.

**REFERENCES**


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