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HARMONIC BLENDING APPROXIMATION

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Dedicated to Professor Werner Haussmann on his sixties birthday

Abstract. The concept of harmonic Hilbert space $H_D(\mathbb{R}^n)$ was introduced in [2] as an extension of periodic Hilbert spaces [1], [2], [5], [6]. In [4] we introduced multivariate harmonic Hilbert spaces and studied approximation by exponential-type function in these spaces and derived error bounds in the uniform norm for special functions of exponential type which are defined by Fourier partial integrals $S_b(f)$:

$$S_b(f)(x) = \int_{\mathbb{R}^n} \chi_{[-b,b]}(t)F(t)\exp(i(t,x))dt,$$

$[-b,b] = [-b_1,b_1] \times ... \times [-b_n,b_n]$, $\quad b_1 > 0, ..., b_n > 0$, where

$$F(t) \sim \left(\tfrac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} f(x)\exp(-i(x,t))dx \ \in L_2(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$$

is the Fourier transform of $f \in L_2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. In this paper we will investigate more general approximation operators S_ψ in harmonic Hilbert spaces of tensor product type.

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1. HARMONIC HILBERT SPACES

The function D is called the *defining function* of the harmonic Hilbert space $H_D(\mathbb{R}^n)$. It satisfies the following conditions:

$$D(-t) = D(t), \quad 0 \le D(t) \le 1, \quad D \in L_1(\mathbb{R}^n) \ (\Rightarrow D \in L_2(\mathbb{R}^n)).$$

The Fourier integral of the defining function is called the *generating function* of the harmonic Hilbert space:

$$d(x) = \int_{\mathbb{R}^n} D(t)\exp(i(x,t))dt \ \in L_2(\mathbb{R}^n).$$

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The generating function is a function from the Wiener algebra $A(\mathbb{R}^n)$. This algebra is defined as the set of functions

$$f(x) = \int_{\mathbb{R}^n} F(t)\exp(i(x,t))dt,$$
$$F(t) = (\tfrac{1}{2\pi})^n \int_{\mathbb{R}^n} f(x)\exp(-i(x,t))dx \in L_1(\mathbb{R}^n).$$

It is a subalgebra of the algebra of uniformly continuous functions on the real line vanishing at infinity $C_0(\mathbb{R}^n)$. The norm of this algebra is the maximum norm: $\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}^n\}$. The norm of the Wiener algebra is given by $\|f\|_a = \int_{\mathbb{R}^n} |F(t)|\,dt$.

The inequality $\|f\|_\infty \leq \|f\|_a$ holds for any function of the Wiener algebra. Note that $F \geq 0$ implies $\|f\|_\infty = \|f\|_a$.

The inner product of the harmonic Hilbert space is defined by

$$(f,g)_D = \int_{\mathbb{R}^n} F(t)\overline{G(t)}\frac{1}{D(t)}dt.$$

It is a reproducing kernel Hilbert space:

$$f(x) = (f, d(\cdot - x))_D.$$

Any harmonic Hilbert space is a subspace of the Wiener algebra:

$$H_D(\mathbb{R}^n) \subseteq A(\mathbb{R}^n) \ \subseteq C_0(\mathbb{R}^n).$$

The imbeddings are continuous due to the estimates

$$\|f\|_\infty \leq \|f\|_a \leq \sqrt{d(0)}\,\|f\|_D\,.$$

Examples of defining functions in the univariate case are taken from summability theory. We give a list of typical examples:

Sobolev space $W^1(\mathbb{R})$:

$$D(t) = \frac{1}{1+t^2}, \quad d(x) = \pi\exp(-|x|),$$

holomorphic Sobolev space $H^1(\mathbb{R})$:

$$D(t) = \exp(-|t|), \quad d(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2},$$

Paley–Wiener space $PW_b(\mathbb{R})$:

$$D(t) = (b - |t|)^0_+, \quad d(x) = 2\sin(bx)/x.$$

Tensor product harmonic Hilbert spaces are obtained by choosing tensor products of univariate defining functions. For notational simplicity we consider mainly the case $n = 2$:

$$D^P(t_1,t_2) = D_1(t_1)D_2(t_2) = (D_1 \otimes D_2)(t_1,t_2),$$
$$d^P(x_1,x_2) = d_1(x_1)d_2(x_2) = (d_1 \otimes d_2)(x_1,x_2),$$
$$H_{D_P}(\mathbb{R}^2) = H_{D_1}(\mathbb{R}) \otimes H_{D_2}(\mathbb{R}).$$

In our examples from the univariate case we use the following notations. Tensor product Sobolev space $W^{(1,1)}(\mathbb{R}^2)$:

$$D(t_1,t_2) = \frac{1}{1+t_1^2}\cdot\frac{1}{1+t_2^2}, \qquad d(x_1,x_2) = \pi^2 \exp(-|x_1|)\exp(-|x_2|).$$

Tensor product holomorphic Sobolev space $H^{(1,1)}(\mathbb{R}^2)$:

$$D(t_1,t_2) = \exp(-|t_1|)\exp(-|t_2|), \qquad d(x_1,x_2) = \frac{1}{\pi^2}\cdot\frac{1}{1+x_1^2}\cdot\frac{1}{1+x_2^2}.$$

Tensor product holomorphic Paley–Wiener space $PW^{(1,1)}_{b_1,b_2}(\mathbb{R}^2)$:

$$D(t_1,t_2) = (b_1-|t_1|)^0_+\ (b_2-|t_2|)^0_+\ , \quad d(x_1,x_2) = 4\cdot\frac{\sin(b_1x_1)}{x_1}\cdot\frac{\sin(b_2x_2)}{x_2}.$$

2. ψ-FOURIER PARTIAL INTEGRALS

We denote the set of functions $\psi \in L_\infty(\mathbb{R}^n)$ satisfying $0 \le \psi(t) \le 1$, $t \in \mathbb{R}^n$, by $L_\infty(\mathbb{R}^n,[0,1])$.

The function $\psi \in L_\infty(\mathbb{R}^n,[0,1])$ is used to define the ψ-Fourier partial integral

$$S_\psi(f)(x) = \int_{\mathbb{R}^n} \psi(t)F(t)\exp(i(x,t))dt$$

as an approximation of the Fourier integral

$$f(x) = S_1(f)(x) = \int_{\mathbb{R}^n} F(t)\exp(i(x,t))dt.$$

The classical Fourier partial integral with respect to the interval $[-b,b]$ is given by the characteristic function

$$\psi(t) = \chi_{[-b,b]}(t),$$
$$S_\psi(f)(x) = \int_{\mathbb{R}^n} \chi_{[-b,b]}(t) F(t) \exp(i(x,t))dt =: S_b(f)(x).$$

In the univariate case we have

$$\psi(t) = \left(1 - \tfrac{|t|}{b}\right)_+^0,$$
$$S_\psi(f)(x) = \int_{-b}^{b} F(t) \exp(ixt)dt =: S_b(f)(x).$$

The Fejér-partial integral is given by

$$\phi(t) = \left(1 - \tfrac{|t|}{b}\right)_+^1,$$
$$S_\phi(f)(x) = \int_{-b}^{b} \left(1 - \tfrac{|t|}{b}\right)_+^1 F(t) \exp(ixt)dt =: F_b(f)(x).$$

S_ψ is a bounded linear operator on $A(\mathbb{R}^n)$. It satisfies the norm inequality

$$\|S_\psi(f)\|_a \leq \|f\|_a .$$

The restriction of S_ψ to the harmonic Hilbert space $H_D(\mathbb{R}^n)$ defines a bounded linear self adjoint operator:

$$(S_\psi(f), g))_D = (S_\psi(f), g))_D .$$

Moreover, the following estimate holds

$$\|S_\psi(f)\|_D \leq \|f\|_D .$$

S_ψ is a projector if and only if

$$\psi^2 = \psi \Leftrightarrow S_\psi^2 = S_\psi.$$

In this case S_ψ is a projector on $A(\mathbb{R}^n)$ and induces by restriction an orthogonal projector on $H_D(\mathbb{R}^n)$.

The approximation order of the ψ-Fourier integral in the harmonic Hilbert space $H_D(\mathbb{R}^n)$ is determined by the remainder of the generating function $d - S_\psi(d)$.

PROPOSITION 2.1. *Assume that $f \in H_D(\mathbb{R}^n)$. Then the error estimate*

$$\|f - S_\psi(f)\|_\infty \le \|f\|_D \, \|1-\psi\|_{2,D}$$

holds with

$$\|d - S_\psi(d)\|_D = \sqrt{\int_{\mathbb{R}^n} (1-\psi(t))^2 D(t) dt} = \|1-\psi\|_{2,D} \, .$$

Proof. The structure of the harmonic Hilbert space as a reproducing kernel Hilbert space implies

$$(f, d(\cdot - x))_D = f(x)$$

in view of

$$d(y-x) = \int_{\mathbb{R}^n} \exp(-i(x,t)) D(t) \exp(i(y,t)) dt$$

and

$$f(x) = \int_{\mathbb{R}^n} \frac{F(t)\overline{D(t)\exp(-i(x,t))}}{D(t)} dt \;\; = (f, d(\cdot - x))_D.$$

Moreover, we have

$$S_\phi(f)(x) = \int_{\mathbb{R}^n} \phi(t) F(t) \exp(i(x,t)) dt$$

which implies

$$\|S_\phi(f)\|_D{}^2 = \int_{\mathbb{R}^n} \phi(t)^2 |F(t)|^2 / D(t) dt.$$

The translation operator and the ψ-Fourier partial integral operator commute:

$$S_\psi(d(\cdot - x))(y) = \int_{\mathbb{R}^n} \psi(t) \exp(-i(x,t)) D(t) \exp(i(y,t)) dt = S_\psi(d)(y-x),$$

i.e., we have

$$S_\psi(d(\cdot - x)) = S_\psi(d)(\cdot - x).$$

Next we can conclude

$$\begin{aligned} f(x) - S_\psi(f)(x) &= (f, d(\cdot - x))_D - (S_\psi(f), d(\cdot - x))_D \\ &= (f, d(\cdot - x))_D - (f, S_\psi(d(\cdot - x)))_D \, , \end{aligned}$$

i.e., we have

$$f(x) - S_\psi(f)(x) = (f, d(\cdot - x) - S_\psi(d(\cdot - x)))_D = (f, S_{1-\psi}(d(\cdot - x)))_D \, .$$

Consider the linear functional on $H_D(\mathbb{R}^n)$ defined by

$$L_{\psi,x}(f) = f(x) - S_\psi(f)(x) = (f, S_{1-\psi}(d(\cdot - x)))_D \, .$$

By the Riezs representation theorem in Hilbert spaces [A.N. Michel, C.J. Herget: Applied Linear Algebra and Functional Analysis] its norm is given by

$$\begin{aligned}\|L_{\psi,x}\| &= \|S_{1-\psi}(d(\cdot - x))\|_D \\ &= \|S_{1-\psi}(d)(\cdot - x)\|_D \\ &= \|S_{1-\psi}(d)\|_D \\ &= \|d - S_\psi(d)\|_D\,.\end{aligned}$$

Since

$$\|d - S_\psi(d)\|_D{}^2 = \int_{\mathbb{R}^n} (1-\psi(t))^2 D(t)dt$$

the proof is complete. □

REMARK 2.1. If S_ψ is an orthogonal projector the sharper estimate

$$\|f - S_\psi(f)\|_\infty \le \|f - S_\psi(f)\|_D \, \|d - S_\psi(d)\|_D$$

holds.

This follows from

$$f(x) - S_\psi(f)(x) = (f, S_{1-\psi}(d(\cdot - x)))_D = (S_{1-\psi}(f), S_{1-\psi}(d(\cdot - x)))_D$$

by an application of the Cauchy–Schwarz inequality .

3. LATTICES OF FOURIER PARTIAL INTEGRAL OPERATORS

We denote the set of functions $\psi \in L_\infty(\mathbb{R}^n)$ satisfying $0 \le \psi(t) \le 1$, $t \in \mathbb{R}^n$, by $L_\infty(\mathbb{R}^n, [0,1])$. We summarize some algebraic properties of $L_\infty(\mathbb{R}^n, [0,1])$:

(1) $\psi \in L_\infty(\mathbb{R}^n, [0,1]) \Rightarrow 1 - \psi \in L_\infty(\mathbb{R}^n, [0,1])$.
(2) $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1]) \Rightarrow \psi \cdot \gamma \in L_\infty(\mathbb{R}^n, [0,1])$.
(3) $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1]) \Rightarrow \psi \oplus \gamma := \psi + \gamma - \psi \cdot \gamma \in L_\infty(\mathbb{R}^n, [0,1])$.
(4) $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1]) \Rightarrow \psi \vee \gamma := \max\{\psi, \gamma\} \in L_\infty(\mathbb{R}^n, [0,1])$.
(5) $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1]) \Rightarrow \psi \wedge \gamma := \min\{\psi, \gamma\} \in L_\infty(\mathbb{R}^n, [0,1])$.
(6) $\psi \cdot \gamma \le \min\{\psi, \gamma\} \le \max\{\psi, \gamma\} \le \psi \oplus \gamma$.

This shows that $L_\infty(\mathbb{R}^n, [0,1])$ is a lattice of real valued measurable functions. The following result is easily verified.

PROPOSITION 3.1. *Assume $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1])$. Then we have*

$$1 - \psi \cdot \gamma = (1-\psi) \oplus (1-\gamma) = (1-\psi) + (1-\gamma) - (1-\psi)\cdot(1-\gamma),$$
$$1 - \psi \oplus \gamma = (1-\psi)\cdot(1-\gamma).$$

The set of commuting non negative Hermitian operators S_ψ forms an operator lattice with respect to the order relation:

$$S_\psi \geq 0 \Leftrightarrow (S_\psi f, f)_D \geq 0, \forall f \in H_D(\mathbb{R}^n).$$

Consider any two functions $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1])$. The associated Fourier partial integral operators S_ψ, S_γ commute and their product is again a *product Fourier partial integral operator* satisfying

$$S_\psi S_\gamma = S_\gamma S_\psi = S_{\psi\gamma}.$$

The Boolean sum of $\psi, \gamma \in L_\infty(\mathbb{R}^n, [0,1])$ defines the *blending Fourier partial integral operator*:

$$S_\psi \oplus S_\gamma = S_\gamma + S_\psi - S_{\psi\gamma}.$$

It is important to note that

$$\begin{aligned} &\psi^2 = \psi \Rightarrow (1-\psi)^2 = 1-\psi \\ &\psi^2 = \psi,\ \gamma^2 = \gamma \ \ \Rightarrow (\psi\gamma)^2 = \psi\gamma,\ (\psi \oplus \gamma)^2 = \psi \oplus \gamma. \end{aligned}$$

Note that the characteristic functions χ_M , χ_N satisfy the above conditions. In particular we have

$$\chi_M \cdot \chi_N = \chi_{M\cap N}, \quad \chi_M \oplus \chi_N = \chi_{M\cup N}, \quad 1 - \chi_M = \chi_{M^c},$$

with $M^c = \mathbb{R}^n - M$.

Proposition 3.2. *The set of operators S_ψ with $\psi^2 = \psi$ form a Boolean algebra of commuting projectors*

$$\mathcal{B} := \{S_\psi : \psi \in L_\infty(\mathbb{R}^n, [0,1]),\ \psi^2 = \psi\}.$$

This aspect turns out to be useful in the multivariate setting.

We first determine the approximation order of product approximation $S_\psi S_\gamma = S_{\psi\cdot\gamma}$.

Proposition 3.3. *Assume that $f \in H_D(\mathbb{R}^n)$. Then the error estimate*

$$\|f - S_\psi S_\gamma(f)\|_\infty \leq \|f\|_D \left(\|1-\psi\|_{2,D} + \|1-\gamma\|_{2,D}\right)$$

holds.

Proof. By Proposition 2.1 we have

$$\|f - S_\psi S_\gamma(f)\|_\infty \leq \|f\|_D \, \|d - S_\psi S_\gamma(d)\|_D \, .$$

Since

$$\|d - S_\psi S_\gamma(d)\|_D = \|d - S_\psi(d) + S_\gamma(d - S_\psi(d))\|_D$$

we obtain

$$\begin{aligned}\|d - S_\psi S_\gamma(d)\|_D &\le \|d - S_\psi(d)\|_D + \|S_\gamma(d - S_\psi(d))\|_D \\ &\le \|d - S_\psi(d)\|_D + \|d - S_\psi(d)\|_D \\ &= \|1 - \psi\|_{2,D} + \|1 - \gamma\|_{2,D} \,.\end{aligned}$$

□

Next we determine the approximation order of blending approximation $S_\psi + S_\gamma - S_{\psi \cdot \gamma} = S_{\psi \oplus \gamma}$.

Proposition 3.4. *Assume that $f \in H_D(\mathbb{R}^n)$. Then the error estimate*

$$\|f - (S_\psi \oplus S_\gamma)(f)\|_\infty \le \|f\|_D \sqrt{\|(1-\psi)^2\|_{2,D}}\sqrt{\|((1-\gamma)^2\|_{2,D}}$$

holds.

Proof. By Proposition 2.1 we have

$$\|f - S_\psi \oplus S_\gamma(f)\|_\infty \le \|f\|_D \, \|d - S_\psi \oplus S_\gamma(d)\|_D$$

Since

$$\begin{aligned}\|S_{1-\gamma\oplus\psi}(d)\|_D{}^2 &= \int_{\mathbb{R}^n} (1 - \gamma(t) \oplus \psi(t))^2 D(t) dt \\ &= \int_{\mathbb{R}^n} (1 - \gamma(t))^2 (1 - \psi(t))^2 D(t) dt \\ &\le \sqrt{\int_{\mathbb{R}^n} (1 - \gamma(t))^4 D(t) dt} \sqrt{\int_{\mathbb{R}^n} (1 - \psi(t))^4 D(t) dt} \\ &= \left\|(1-\psi)^2\right\|_{2,D} \cdot \left\|(1-\gamma)^2\right\|_{2,D}\end{aligned}$$

the proof is complete. □

As a special case we obtain

Proposition 3.5. *Assume that $f \in H_D(\mathbb{R}^n)$. Then the error estimate*

$$\|f - (S_\psi \oplus S_\psi)(f)\|_\infty \le \|f\|_D \left\|(1-\psi)^2\right\|_{2,D}$$

holds.

4. APPROXIMATION IN TENSOR PRODUCT HARMONIC HILBERT SPACES

In the tensor product harmonic Hilbert space

$$H_{D^P}(\mathbb{R}^2) = H_D(\mathbb{R}) \otimes H_D(\mathbb{R})$$

we have the simple situation

$$D^P(t_1,t_2) = D(t_1)D(t_2) = D\otimes D(t_1,t_2),$$
$$d^P(x_1,x_2) = d(x_1)d(x_2) = d\otimes d(x_1,x_2).$$

This leads to special constructions of ψ-Fourier integrals choosing tensor products of measurable functions:

$$\gamma(t_1,t_2) = \phi(t_1) = \phi\otimes 1_{\mathbb{R}}(t_1,t_2),$$
$$\psi(t_1,t_2) = \zeta(t_2) = 1_{\mathbb{R}}\otimes\zeta(t_1,t_2).$$

Recall that

$$\|d - S_\psi(d)\|_D = \sqrt{\int_{\mathbb{R}^n}(1-\psi(t))^2 D(t)dt} = \|1-\psi\|_{2,D}\,.$$

The tensor product structure implies

$$\begin{aligned}
\left\|d^P - S_{\phi\otimes 1_{\mathbb{R}}}(d^P)\right\|_{D^P} &= \left\|S_{(1_{\mathbb{R}}-\phi)\otimes 1_{\mathbb{R}}}(d^P)\right\|_{D^P}\\
&= \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1-\phi(t_1))^2 dt_1 dt_2}\\
&= \|1-\phi\|_{2,D}\|1\|_{2,D},\\
\left\|d^P - S_{1_{\mathbb{R}}\otimes\zeta}(d^P)\right\|_{D^P} &= \left\|S_{1_{\mathbb{R}}\otimes(1_{\mathbb{R}}-\zeta)}(d^P)\right\|_{D^P}\\
&= \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1-\zeta(t_2))^2 dt_1 dt_2}\\
&= \|1-\zeta\|_{2,D}\|1\|_{2,D}.
\end{aligned}$$

An application of Proposition 3.3 yields

Proposition 4.1. *Assume that $f\in H_D(\mathbb{R})\otimes H_D(\mathbb{R})$. Then the error estimate*

$$\|f - S_{\phi\otimes\zeta}(f)\|_\infty \le \|f\|_{D^P}\,\|1\|_{2,D}\left(\|1-\phi\|_{2,D} + \|1-\zeta\|_{2,D}\right)$$

holds.

For blending approximation in $H_{D_P}(\mathbb{R}^2)$ with the operator

$$S_{\phi\otimes 1_{\mathbb{R}}}\oplus S_{1_{\mathbb{R}}\otimes\zeta} = S_{1_{\mathbb{R}}\otimes\zeta} + S_{\phi\otimes 1_{\mathbb{R}}} - S_{\phi\otimes\zeta}$$

it follows

$$\begin{aligned}
&\left\|d^P - S_{\phi\otimes 1_{\mathbb{R}}}\oplus S_{1_{\mathbb{R}}\otimes\zeta}(d^P)\right\|_{D^P} =\\
&\quad= \left\|S_{(1_{\mathbb{R}}-\phi)\otimes(1_{\mathbb{R}}-\zeta)}(d^P)\right\|_{D^P}\\
&\quad= \sqrt{\int_{\mathbb{R}^2} D(t_1)D(t_2)(1-\phi(t_1))^2(1-\zeta(t_2)^2 dt_1 dt_2}\\
&\quad= \|1-\phi\|_{2,D}\|(1-\zeta\|_{2,D}.
\end{aligned}$$

Thus we have shown

PROPOSITION 4.2. *Assume that* $f \in H_D(\mathbb{R}) \otimes H_D(\mathbb{R})$. *Then the error estimate*

$$\|f - S_{\phi\otimes 1_{\mathbb{R}}} \oplus S_{1_{\mathbb{R}}\otimes\zeta}(f)\|_\infty \le \|f\|_{D^P} \,\|1-\phi\|_{2,D}\|1-\zeta\|_{2,D}$$

holds.

5. EXPONENTIAL-TYPE BLENDING APPROXIMATION

As a classical example let

$$\gamma(t_1,t_2) = \chi_{[-b_1,b_1]}(t_1),\ \ \psi(t_1,t_2) = \chi_{[-b_2,b_2]}(t_2),$$

In this case the operators are parametrically extended univariate Fourier partial integrals

$$S_{b_1}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_1,b_1]}(t_1)F(t)\exp(i(x,t))dt,$$
$$S_{b_2}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_2,b_2]}(t_2)F(t)\exp(i(x,t))dt$$

These are functions of exponential-type in x_1, respectively in x_2. The corresponding product operator $S_{b_1}S_{b_2}$ is the bivariate Fourier partial integral

$$S_{b_2}S_{b_1}(f)(x) = \int_{\mathbb{R}^2} \chi_{[-b_1,b_1]}(t_1)\chi_{[-b_2,b_2]}(t_2)F(t)\exp(i(x,t))dt = S_{(b_1,b_2)}(f)(x).$$

These are bivariate functions of exponential-type.
Recall that

$$\|d - S_\phi(d)\|_D = \sqrt{\int_{\mathbb{R}^n} (1-\phi(t))^2 D(t)dt} = \|1-\phi\|_{2,D}\,.$$

PROPOSITION 5.1. *The asymptotic error estimate for the bivariate Fourier partial integral follows from the general result Proposition* 4.1

$$\left\|f - S_{(b_1,b_2)}(f)\right\|_\infty = \mathcal{O}\left(\sqrt{\int_{b_1}^\infty D(t)dt} + \sqrt{\int_{b_2}^\infty D(t)dt}\right),\ b_1 \wedge b_2 \uparrow \infty.$$

The Boolean sum $S_{b_1} \oplus S_{b_2}$ is called the bivariate hyperbolic cross Fourier integral

$$\begin{aligned} &S_{b_2} \oplus S_{b_1}(f)(x) = \\ &= \int_{\mathbb{R}^2} \chi_{[-b_1,b_1]}(t_1)F(t)\exp(i(x,t))dt + \int_{\mathbb{R}^2} \chi_{[-b_2,b_2]}(t_2)F(t)\exp(i(x,t))dt \\ &\quad - \int_{\mathbb{R}^2} \chi_{[-b_1,b_1]}(t_1)\chi_{[-b_2,b_2]}(t_2)F(t)\exp(i(x,t))dt \ =: S^{(b_1,b_2)}(f)(x). \end{aligned}$$

These are bivariate blending functions of exponential-type.

Proposition 5.2. *The asymptotic error estimate for the bivariate hyperbolic cross Fourier integral follows from the general result Proposition* 4.2

$$\left\|f - S^{(b_1,b_2)}(f)\right\|_\infty = \mathcal{O}\left(\sqrt{\int_{b_1}^{\infty} D(t)dt} \cdot \sqrt{\int_{b_2}^{\infty} D(t)dt}\right), \quad b_1 \vee b_2 \uparrow \infty.$$

The first example is the tensor product Sobolev space. The univariate defining function is

$$D(t) = \frac{1}{1+t^2}$$

and it defines the univariate Sobolev space

$$H_D(\mathbb{R}) = W^1(\mathbb{R}),$$

The bivariate tensor product harmonic Hilbert space is obtained by choosing the tensor product defining function,

$$H_{D_P}(\mathbb{R}^2) = W^{(1,1)}(\mathbb{R}^2).$$

The univariate error norms satisfy the asymptotic relations

$$\sqrt{\int_{b_1}^{\infty} \tfrac{1}{1+t^2} dt} = \mathcal{O}\Big(b_1^{-\frac{1}{2}}\Big), \qquad \sqrt{\int_{b_2}^{\infty} \tfrac{1}{1+t^2} dt} = \mathcal{O}\Big(b_2^{-\frac{1}{2}}\Big).$$

Corollary 5.1. *The asymptotic error estimate for the bivariate product Fourier partial integral in* $W^{(1,1)}(\mathbb{R}^2)$ *is given by*

$$\left\|f - S_{(b_1,b_2)}(f)\right\|_\infty = \mathcal{O}\Big(b_1^{-\frac{1}{2}} + b_2^{-\frac{1}{2}}\Big), \qquad b_1 \wedge b_2 \uparrow \infty$$

It is determined by the maximal univariate error bound.

Corollary 5.2. *The asymptotic error estimate for the bivariate hyperbolic cross Fourier partial integral in* $W^{(1,1)}(\mathbb{R}^2)$ *is given by*

$$\left\|f - S^{(b_1,b_2)}(f)\right\|_\infty = \mathcal{O}\left((b_1 b_2)^{-\frac{1}{2}}\right), \quad b_1 \vee b_2 \uparrow \infty.$$

It is determined by the product of the univariate error bounds.

For the univariate defining function

$$D(t) = \exp(-|t|)$$

the bivariate tensor product holomorphic Sobolev space is obtained,

$$H_{D_P}(\mathbb{R}^2) = H^{(1,1)}(\mathbb{R}^2).$$

The univariate error norms satisfy the asymptotic relations

$$\sqrt{\int_{b_1}^{\infty} \exp(-|t|)dt} = \mathcal{O}\Big(\exp(-\tfrac{b_1}{2})\Big), \qquad \sqrt{\int_{b_2}^{\infty} \exp(-|t|)dt} = \mathcal{O}\Big(\exp(-\tfrac{b_2}{2})\Big).$$

COROLLARY 5.3. *The asymptotic error estimate for the bivariate product Fourier partial integral in* $H^{(1,1)}(\mathbb{R}^2)$ *is given by*

$$\left\| f - S_{(b_1,b_2)}(f) \right\|_\infty = \mathcal{O}\left(\exp(-\tfrac{b_1}{2}) + \exp(-\tfrac{b_2}{2})\right), \qquad b_1 \wedge b_2 \uparrow \infty.$$

It is determined by the maximal univariate error bound.

COROLLARY 5.4. *The asymptotic error estimate for the bivariate hyperbolic cross Fourier partial integral in* $H^{(1,1)}(\mathbb{R}^2)$ *is given by*

$$\left\| f - S^{(b_1,b_2)}(f) \right\|_\infty = \mathcal{O}\left(\exp(-\tfrac{b_1+b_2}{2})\right), \quad b_1 \vee b_2 \uparrow \infty.$$

It is determined by the product of the univariate error bounds.

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