REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 30 (2001) no. 2, pp. 163–178
ictp.acad.ro/jnaat

DEGREE OF SIMULTANEOUS APPROXIMATION BY BIRKHOFF SPLINES

HEINER GONSKA* and DANIELA KACSÓ†

Dedicated to Professor Werner Haußmann on his sixtieth birthday.

Abstract. In the present note we study the degree of simultaneous approximation by certain Birkhoff spline interpolation operators. Special emphasis is on estimates in terms of higher order moduli of smoothness. This generalizes earlier results of Meir and Sharma, Demko, Howell and Varma, and Buckett and Varma.

MSC 2000. 41A05, 41A15, 41A25.

1. INTRODUCTION AND PROBLEM DESCRIPTION

The present paper is dedicated to Professor Werner Haußmann who has had a significant impact on the professional careers of both authors. Professor Haußmann has constantly shown an interest in interpolation, including lacunary (Birkhoff) interpolation by polynomials and other function spaces. Likewise he encouraged several of his students to consider the problems of simultaneous and quantitative approximation. The present paper deals with certain spline cases. Our focus is on error estimates for simultaneous approximation in terms of higher order moduli of smoothness. All results are based on a powerful general estimation technique which turned out to be rather useful in [6], [10] and [11].

In a 1968 paper I.J. Schoenberg [16] initiated the study of so-called g-splines in connection with the problem of lacunary interpolation by splines (see [13]). It seems, however, that the 1973 article by Meir and Sharma gave even more impetus to the further development of the theory of lacunary (Birkhoff) splines. In this paper–in which the spline <u>knots</u> and the interpolation <u>nodes</u>

*FB 11–Informatik I, Gerhard Mercator University, D–47048 Duisburg, Germany, e-mail: gonska@informatik.uni-duisburg.de.

†FB 11–Informatik I, Gerhard Mercator University, D–47048 Duisburg, Germany, e-mail: kacso@informatik.uni-duisburg.de, on leave from: Department of Numerical and Statistical Calculus, Faculty of Mathematics and Computer Science, Babeş–Bolyai University, RO–3400 Cluj–Napoca, Romania.

coincide—they also indicate that it might be of interest to investigate the analogous problems when this is not the case (see [13, p. 442]). This was done indeed in several later papers, some of which we will also discuss here.

As in the polynomial case incidence matrices E are useful to visualize an individual interpolation problem. Let $a \leq y_0 < y_1 < \ldots < y_m \leq b$ be a sequence of arbitrary points in the interval $[a, b]$. With this sequence of points we associate a matrix

$$E = (e_{i,j}) \;\; i = 0, \ldots, m; \;\; j = 0, \ldots, R,$$

where R is a positive integer.

Such matrices have as entries $|E| \geq m + 1$ ones and $(m + 1)(R + 1) - |E|$ zeros and are such that in each row (corresponding to one of the y_i's) there exists at least one entry equal to one. We also assume that the last column contains at least one entry equal to one.

The partition $y_0 < y_1 < \ldots < y_m$ constitutes the union of the sets of spline knots $\{x_k\}$ (which are also interpolation nodes) and of further interpolation nodes $\{z_l\}$. A typical case is

$$x_0 < z_0 < x_1 < z_1 < \ldots < x_{n-1} < z_{n-1} < x_n;$$

so here

$$\begin{aligned} y_0 = x_0 < y_1 = z_0 < y_2 = x_1 < y_3 = z_1 < \ldots \\ \ldots < y_{2n-2} = x_{n-1} < y_{2n-1} = z_{n-1} < y_{2n} = x_n. \end{aligned}$$

The set $\{z_l\}$ may be empty, that is, the interpolation nodes may coincide with the spline knots.

Lacunary spline interpolation consists of finding a spline s (sufficiently smooth and to be specified below) such that the following $|E|$ conditions are fulfilled:

$$s^{(j)}(y_i) \;\; = \;\; a_i^{(j)} \quad \text{if } e_{i,j} = 1.$$

Here the $a_i^{(j)}$ are arbitrary real numbers.

We will not deal with the problems of existence and uniqueness of Birkhoff spline interpolation here. A valuable source of information in regard to these is the book by Lorentz, Jetter and Riemenschneider [12].

In the present note we will only consider cases in which—assuming unique solutions—for $f \in C^R[a, b]$ we put

$$a_i^{(j)} \;\; = \;\; f^{(j)}(y_i) \quad \text{if } e_{i,j} = 1.$$

Since the last column of E has at least one entry equal to one, $f^{(R)}(y_i)$ is indeed needed for some $i \in \{0, \dots, m\}$ in this setting.

One further case of interest is the one in which $f \in C^{R'}[a, b]$ with $0 \le R' < R$. In this situation one typically requires

$$a_i^{(j)} = 0 \quad \text{for } R' + 1 \le j \le R \text{ and } e_{i,j} = 1.$$

For details in the context of Birkhoff interpolation by polynomials see, e.g., [10], [11].

Forcing some $a_i^{(j)}$ to be equal to zero takes us into quite a different situation which one might name "Birkhoff–Fejér interpolation by spline functions". In fact, this was also considered in the spline case, but we will not deal with this question here. See [8] for many references on how this problem was dealt with in the polynomial case.

In the sequel we study the degree of uniform approximation of $f \in C^R[a, b]$ and its derivatives up to a certain order by interpolating splines in the Schoenberg spaces $\mathcal{S}_{\Delta,q}^{(r)}$ and their corresponding derivatives.

Thus $\mathcal{S}_{\Delta,q}^{(r)}$, $-1 \le r < q$, is the class of splines s such that

(i) $s \in C^r[0, 1]$,
(ii) $s \in \Pi_q$ on $[x_\nu, x_{\nu+1}]$, $0 \le \nu \le n - 1$, where $\Delta = \Delta_n : a = x_0 < x_1 < \dots < x_n = b$ are the spline knots.

We recall that $\mathcal{S}_{\Delta,q}^{(-1)}$ are the splines which may be discontinuous at x_i, $0 \le i \le n$, and that the number $k = q - r \ge 1$ is the defect of the splines. More on (more general) Schoenberg spaces can be found in [5, Ch. 5], for example.

As usual we put $h = h_n := \max\{x_{\nu+1} - x_\nu : 0 \le \nu \le n - 1\}$; this is the "mesh gauge". We will thus omit the subscript n if it is clear from the context.

The "mesh ratio" is given by

$$\beta_n = \frac{\max\{x_{\nu+1} - x_\nu\}}{\min\{x_{\nu+1} - x_\nu\}}.$$

The only function norm used throughout this note is the sup–norm $\|\cdot\|_\infty$, we will thus simply denote it by $\|\cdot\|$.

All inequalities below will be given in terms of higher order moduli of smoothness $\omega_s(f;\delta)$. For $f \in C[a, b]$, $s \in \mathbb{N}_0$, $\delta \ge 0$, the latter quantities are defined by

$$\omega_s(f;\delta) := \sup\{|\Delta_h^s f(x)| : x, x + sh \in [a, b], |h| \le \delta\},$$

where

$$\Delta_h^s f(x) := \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} f(x+\nu h).$$

Properties of higher order moduli are collected in L. Schumaker's book [17, p. 55f.], for example. We mention just one of them, which is important as an explanation for why our estimates given below imply what was already known, namely:

$$\text{if} \quad f \in C^r[a,b], \ \ r \in \mathbb{N}_0, \quad \text{then } \omega_{s+r}(f;\delta) \le \delta^r \cdot \omega_s(f^{(r)},\delta), \ \delta \ge 0.$$

2. MAIN RESULT

The approach via higher order moduli was also used in [7] where refined inequalities where given for the so–called “clamped” cubic splines as considered by Sharma and Meir [15], among others. An essential tool there, and also in our other papers on Birkhoff interpolation, was the following lemma first given in [6]. It describes how well a function $f \in C^r(I)$ can be smoothed by certain smoother functions.

Lemma 2.1. *Let* $I = [0,1]$, $f \in C^r(I)$, $r \in \mathbb{N}_0$. *For any* $0 < \delta \le 1$ *and* $s \in \mathbb{N}$ *there exists a function* $f_{\delta,r+s} \in C^{2r+s}(I)$ *with*

(i) $\|f^{(j)} - f_{\delta,r+s}^{(j)}\| \le c \cdot \omega_{r+s}(f^{(j)},\delta), \ \ 0 \le j \le r;$
(ii) $\|f_{\delta,r+s}^{(j)}\| \le c \cdot \delta^{-j} \cdot \omega_j(f,\delta), \ \ 0 \le j \le r+s;$
(iii) $\|f_{\delta,r+s}^{(j)}\| \le c \cdot \delta^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)},\delta), \ \ r+s \le j \le 2r+s.$

Here the constant c *depends only on* r *and* s.

As was noted in [6], the statement of the lemma can be carried over to any finite interval $[a,b]$, $a < b$, by using the suitable linear transformation, and the impact of this transformation will only be on the constant c figuring in the lemma. In the sequel c will denote a constant that can be different at every occurence, even in the same formula.

In the following theorem we adjust the smoothing approach also used earlier to the present situation. The quantities $k_u \le p_l \le p_u$ figuring there stand for “$k-$upper”, “$p-$lower” “$p-$upper”, respectively. The operators S_n in the theorem are indexed only in order to have a subscript available also indicating the dependence of certain constants.

Theorem 2.2. *Let $0 \le k_u \le p_l \le p_u$ be given integers. For $n \in \mathbb{N}$, let $S_n : C^{p_l}[a,b] \to C^{k_u}[a,b]$ be a linear operator such that the following hold: For $p \in \{p_l, p_u\}$ and $g \in C^p[a,b]$ we have*

$$\|(S_n g - g)^{(k)}\| \le \varepsilon_{p,k} \cdot \delta_n(p,k)\, \|g^{(p)}\|, \quad 0 \le k \le k_u,$$

where the constants $\varepsilon_{p,k}$ are independent of n, but $\delta_n(p,k)$ depends on all three parameters.

Then for all $f \in C^{p_l}[a,b],\ 0 \le k \le k_u$ and any $0 < \delta \le b-a$,

$$\|(S_n f - f)^{(k)}\| \le c \cdot \left[\delta_n(p_l,k) + \delta_n(p_u,k) \cdot \delta^{-(p_u-p_l)}\right] \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta).$$

Here the constant c depends on k_u, p_l, p_u, and all the $\varepsilon_{p,k}$.

Proof. For $f \in C^{p_l}$, $g \in C^{p_u}$ and $0 \le k \le k_u$ we write

$$\begin{aligned} \|(S_n f - f)^{(k)}\| &\le \|[S_n(f-g) - (f-g)]^{(k)}\| + \|(S_n g - g)^{(k)}\| \\ &\le \varepsilon_{p_l,k} \cdot \delta_n(p_l,k) \cdot \|(f-g)^{(p_l)}\| + \varepsilon_{p_u,k} \cdot \delta_n(p_u,k) \cdot \|g^{(p_u)}\|. \end{aligned}$$

For $p_l = p_u$ put $g = f$ to obtain the original inequality. If $p_l < p_u$ and $0 < \delta \le b-a$, it follows from Lemma 2.1 (use $r = 0$ and $s = p_u - p_l$ there) that there exists a function $g_{\delta,p_u-p_l} \in C^{p_u-p_l}[a,b]$ such that

$$\|f^{(p_l)} - g_{\delta,p_u-p_l}\| \quad \le \quad c \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta).$$

If G_δ denotes a p_l–th primitive of g_{δ,p_u-p_l}, then $G_\delta \in C^{p_u}[a,b]$, and the latter inequality becomes

$$\|f^{(p_l)} - G_\delta^{(p_l)}\| \quad \le \quad c \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta).$$

Furthermore, Lemma 2.1 also shows that

$$\|G_\delta^{(p_u)}\| = \|G_\delta^{(p_l+p_u-p_l)}\| = \|g_\delta^{(p_u-p_l)}\| \le c \cdot \delta^{-(p_u-p_l)} \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta).$$

Hence, for $0 \le k \le k_u$, one has

$$\begin{aligned} &\|(S_n f - f)^{(k)}\| \le \\ &\le \max\{\varepsilon_{p_l,k};\varepsilon_{p_u,k}\} \cdot \Big[\delta_n(p_l,k) \cdot c \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta) + \\ &\qquad\qquad + \delta_n(p_u,k) \cdot c \cdot \delta^{-(p_u-p_l)} \cdot \omega_{p_u-p_l}(f^{(p_l)};\delta)\Big] = \end{aligned}$$

$$= c \cdot \max\{\varepsilon_{p_l,k};\varepsilon_{p_u,k}\} \cdot \left[\delta_n(p_l,k) + \delta_n(p_u,k)\cdot \delta^{-(p_u-p_l)}\right] \omega_{p_u-p_l}(f^{(p_l)};\delta).$$

From this we obtain the claim of the theorem. □

3. THE QUINTIC MEIR–SHARMA LACUNARY INTERPOLANT – MODIFIED (0,2) CASE

In a 1973 paper by A. Meir and A. Sharma [13], error bounds were given for lacunary interpolation of certain functions by *deficient* quintic splines. We recall that $\mathcal{S}_{\Delta,5}^{(3)}$ denotes the class of quintic splines s such that

(i) $s \in C^3[0,1]$,
(ii) $s \in \Pi_5$ on each interval $\left[\frac{\nu}{n}, \frac{\nu+1}{n}\right]$, $0 \le \nu \le n-1$.

Given $f \in C^3[0,1]$, for n odd, let s_n be the *unique* element (cf. [13, Theorem 1]) in $\mathcal{S}_{\Delta,5}^{(3)}$ which interpolates f in the sense that

(i) $s_n\left(\frac{\nu}{n}\right) = f\left(\frac{\nu}{n}\right)$, $0 \le \nu \le n$;
(ii) $s_n''\left(\frac{\nu}{n}\right) = f''\left(\frac{\nu}{n}\right)$, $0 \le \nu \le n$;
(iii) $s_n'''(0) = f'''(0)$, $s_n'''(1) = f'''(1)$.

The interpolant is sometimes called the Meir–Sharma interpolant of f. Its interpolation conditions can be visualized by the following scheme (incidence matrix) from which it is clear that we are dealing with a modified $(0,2)$–interpolation problem here ($x_\nu = \frac{\nu}{n}$, $0 \le \nu \le n$):

$$\begin{array}{lcccc}
x_0 & 1 & 0 & 1 & 1 \\
x_1 & 1 & 0 & 1 & 0 \\
x_2 & 1 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-1} & 1 & 0 & 1 & 0 \\
x_n & 1 & 0 & 1 & 1
\end{array}$$

Meir and Sharma showed that, for $f \in C^4[0,1]$,

$$\|(f-s_n)^{(k)}\| \le 75 \cdot \frac{1}{n^{3-k}} \cdot \omega_1\left(f^{(4)};\tfrac{1}{n}\right) + 8 \cdot \frac{1}{n^{4-k}}\|f^{(4)}\|,\ 0 \le k \le 3.$$

Their lacunary interpolant was further investigated (and generalized) in a note by B.K. Swartz and R.S. Varga [18]. The latter mentioned authors showed in Theorem 1 in [18], among other things, the following:

(i) for $f \in C^3[0,1]$,

$$\begin{aligned}\|(f-s_n)^{(k)}\| &\leq \varepsilon_{3,k} \cdot \frac{1}{n^{2-k}} \cdot \|f^{(3)}\| \\ &=: \varepsilon_{3,k} \cdot \delta_n(3,k) \cdot \|f^{(3)}\|, \quad 0 \leq k \leq 3,\end{aligned}$$

with $\varepsilon_{3,k}$ independent of f and n.

Furthermore, in their Lemma 1 they proved

(ii) for $f \in C^6[0,1]$,

$$\begin{aligned}\|(f-s_n)^{(k)}\| &\leq 20 \cdot \frac{1}{n^{5-k}} \cdot \|f^{(6)}\| \\ &=: 20 \cdot \delta_n(6,k) \cdot \|f^{(6)}\|, \quad 0 \leq k \leq 3.\end{aligned}$$

Thus all the information is available to apply Theorem 2.2 in order to arrive at

Proposition 3.1. *Let s_n be the Meir–Sharma lacunary interpolant. Then, for any $f \in C^3[0,1]$,*

$$\|(f-s_n)^{(k)}\| \leq c \cdot \frac{1}{n^{2-k}} \cdot \omega_3\left(f'''; \tfrac{1}{n}\right), \quad 0 \leq k \leq 3,$$

with an absolute constant c.

Proof. In Theorem 2.2 we put $k_u = 3,\ p_l = 3,\ p_u = 6$. This yields

$$\|(f-s_n)^{(k)}\| \leq c \cdot \left[\frac{1}{n^{2-k}} + \frac{1}{n^{5-k}} \cdot \delta^{-3}\right] \cdot \omega_3(f'''; \delta),$$

where $0 < \delta \leq 1$ is arbitrary. Putting $\delta = \frac{1}{n}$ gives

$$\|(f-s_n)^{(k)}\| \leq c \cdot \frac{1}{n^{2-k}} \cdot \omega_3\left(f'''; \tfrac{1}{n}\right), \quad 0 \leq k \leq 3.$$

□

Corollary 3.2. *For $f \in C^4[0,1]$, the estimate in Proposition 3.1 gives*

$$\|(f-s_n)^{(k)}\| \leq c \cdot \frac{1}{n^{3-k}} \cdot \|f^{(4)}\|, \quad 0 \leq k \leq 3,$$

and for $f \in C^5[0,1]$ we get

$$\|(f-s_n)^{(k)}\| \leq c \cdot \frac{1}{n^{4-k}} \cdot \|f^{(5)}\|, \quad 0 \leq k \leq 3.$$

Thus the original asymptotic order in the estimate of Meir and Sharma is fully contained in the more elegant statement of Proposition 3.1. This is also true for $f^{(4)} \in \mathrm{Lip}_\alpha[0,1]$, a case mentioned explicitly by them.

REMARK 3.3. Swartz and Varga also gave estimates for $k = 4$ if $f \in C^4[0,1]$. Since s_n is piecewise quintic and in $C^3[0,1]$ only, between the interpolation knots $s_n^{(4)}$ is a linear function and possibly an element of $L_\infty[0,1] \setminus C[0,1]$. This situation is not covered by Theorem 2.2, and an extension of it is needed with $C^{k_u}[a,b]$ replaced by, say,

$$W_{k_u+1,\infty}[a,b] = \{f \in C^{k_u}[a,b] \,:\, f^{(k_u+1)} \text{ exists a.e. and is in } L_\infty[a,b]\}.$$

4. DEMKO'S GENERALIZED LACUNARY SPLINE INTERPOLANTS

The work of Meir and Sharma discussed in the previous section was remarkably generalized in a 1976 article of S. Demko [4]. We will show next that part of his results can also be improved using higher order moduli of smoothness. Again we are dealing with uniform partitions here. In order to formulate our result, we first cite

THEOREM 4.1. (see [4, Theorem 2.4]). *Let $\Delta : a = x_0 < \ldots < x_n = b$ be a uniform partition of $[a,b]$ with $n > 2$. Let A, B, C be disjoint sets such that $1 \le |A| = d < q$, $A \cup B \cup C = \{0, 1, \ldots, 2q-d-1\}$, $|B| = |C| = q-d$ and $j+k$ is even for $(j,k) \in B \times C$ and $0 \in A \cup B$. Then given arbitrary real data $\{f_{i,j} : j \in A,\ 0 \le i \le n\}$, $\{g_{i,j} : j \in B,\ i = 0, n\}$, there is a unique element $s \in \mathcal{S}_{\Delta,2q-1}^{(2q-d-1)}$ satisfying*

(i) $s^{(j)}(x_i) = f_{i,j},\ \ j \in A,\ \ 0 \le i \le n;$
(ii) $s^{(j)}(x_i) = g_{i,j},\ \ j \in B,\ \ i = 0, n,$

if and only if n is odd.

The existence and uniqueness result of Meir and Sharma is obtained for the special case $A = \{0,2\},\ B = \{3\},\ C = \{1\}$, where $d = 2,\ q = 3$, so that the spline space in question is indeed $\mathcal{S}_{\Delta,5}^{(3)}$.

Demko also proved (an even stronger form of)

THEOREM 4.2. (see [4, Theorem 3.3]). *Let A, B, C and Δ be as in Theorem 4.1. Given $f \in C^{2q}[a,b]$, let s be the unique element in $\mathcal{S}_{\Delta,2q-1}^{(2q-d-1)}$ interpolating f in the following sense:*

$$\begin{aligned} s^{(j)}(x_i) &= f^{(j)}(x_i),\ \ j \in A,\ \ 0 \le i \le n; \\ s^{(j)}(x_i) &= f^{(j)}(x_i),\ \ j \in B,\ \ i = 0, n. \end{aligned}$$

Then

$$\|(s-f)^{(k)}\| \le K \cdot h^{2q-1-k} \cdot \|f^{(2q)}\|, \qquad 0 \le k \le 2q-d-1.$$

Another of his results was (a more general form of)

COROLLARY 4.3. (see [4, Corollary 3.4]). *Let A, B, C and Δ be as in Theorem* 4.1. *Let $f \in C^p[a,b], 0 \le p < 2q-1$. Suppose that* $\max\{l : l \in A \cup B\} \le p$. *Let s be the unique element of $\mathcal{S}^{(2q-d-1)}_{\Delta,2q-1}$ satisfying*

$$\begin{aligned} s^{(j)}(x_i) &= f^{(j)}(x_i), \ \ j \in A, \ \ 0 \le i \le n; \\ s^{(j)}(x_i) &= f^{(j)}(x_i), \ \ j \in B, \ \ i = 0, n. \end{aligned}$$

Then

$$\begin{aligned} \|(s-f)^{(k)}\| &\le K \cdot h^{p-1-k} \cdot \omega(f^{(p)}; h) \\ &\le 2K \cdot h^{p-1-k} \cdot \|f^{(p)}\|, \ \ 0 \le k \le \min\{p, 2q-d-1\}. \end{aligned}$$

We now apply Theorem 2.2 to the cases considered above. We then get

PROPOSITION 4.4. *Let A, B, C and Δ be as in Theorem* 4.1. *Let $f \in C^p[a,b]$, $0 \le p < 2q-1$. Suppose that* $\max\{l : l \in A \cup B\} \le p$, *and let s be the unique spline considered above. Then we have*

$$\|(s-f)^{(k)}\| \le c \cdot h^{p-1-k} \cdot \omega_{2q-p}(f^{(p)}; h), \ \ 0 \le k \le \min\{p, 2q-d-1\}.$$

Proof. In Theorem 2.2 we set $k_u = \min\{p, 2q-d-1\}$, $p_l = p$, $p_u = 2q$, $\delta_n(p,k) = h^{p-1-k}$, $\delta_n(2q,k) = h^{2q-1-k}$, $\delta = h$. This gives

$$\begin{aligned} \|(s-f)^{(k)}\| &\le c \cdot \left[h^{p-1-k} + h^{2q-1-k} \cdot h^{-(2q-p)}\right] \cdot \omega_{2q-p}(f^{(p)}; h) \\ &= c \cdot h^{p-1-k} \cdot \omega_{2q-p}(f^{(p)}; h), \qquad 0 \le k \le \min\{p, 2q-d-1\}. \end{aligned}$$

□

REMARK 4.5. In case of the Meir–Sharma interpolant the parameters are as follows:

$$\begin{aligned} &\max\{l : l \in A \cup B\} = 3 = p, \\ &\min\{p, 2q-d-1\} = \min\{3,3\} = 3, \\ &2q - p = 3. \end{aligned}$$

We have thus rediscovered Proposition 3.1 as a special case of Proposition 4.4.

One further example explicitely mentioned in Demko's paper is an interpolation scheme investigated by Carlson and Hall in [3] and called "Scheme C" there. Using Demko's notation, this is the case $A = \{1\}$, $B = \{0\}$, $C = \{2\}$, and is visualized in the interpolation matrix below:

$$\begin{array}{ccc}
x_0 & 1 & 1 \\
x_1 & 0 & 1 \\
x_2 & 0 & 1 \\
\vdots & \vdots & \vdots \\
x_{n-1} & 0 & 1 \\
x_n & 1 & 1
\end{array}$$

Here the parameters are

$$\begin{aligned}
\max\{l : l \in A \cup B\} &= 1 \le p < 3 = 2q-1, \\
\min\{p, 2q-d-1\} &= \min\{p, 4-1-1\} = \begin{cases} 1, & \text{if } p=1 \\ 2, & \text{if } p=2 \end{cases}, \\
2q-p = 4-p &= \begin{cases} 3, & \text{if } p=1 \\ 2, & \text{if } p=2 \end{cases}.
\end{aligned}$$

Proposition 4.4 implies

PROPOSITION 4.6. *Let $f \in C^p[a,b]$, $p=1$ or $p=2$. Then the "Scheme C" interpolant $s \in \mathcal{S}^{(2)}_{\Delta,3}$ satisfies the inequalities:*

$$\|(s-f)^{(k)}\| \le c \cdot h^{p-1-k} \cdot \omega_{4-p}(f^{(p)}; h), \quad 0 \le k \le \min\{p, 2\}.$$

In particular, if $f \in C^4[a,b]$, then

$$\begin{aligned}
\|(s-f)^{(k)}\| &\le c \cdot h^{1-k} \cdot \omega_2(f''; h) \\
&\le c \cdot h^{3-k} \cdot \|f^{(4)}\|, \quad 0 \le k \le 2.
\end{aligned}$$

REMARK 4.7. As was already mentioned by Demko (see [4, p. 375]), Carlson and Hall (see Corollary 3 in [3]), proved an order of $\mathcal{O}(h^4)$ for $f \in C^5[a,b]$ and $k=0$. The general method from above does not provide this order. This is due to the fact that, for the present value of $q=2$, Theorem 4.2 gives $\mathcal{O}(h^3)$ for $f \in C^4[a,b]$, and no more than that. This calls for a refinement of Demko's method in order to provide subtler input for smooth functions.

5. QUARTIC HOWELL–VARMA INTERPOLANTS–MODIFIED (0,2) CASE

G. Howell and A. Varma [9] considered deficient quartic lacunary interpolants in $\mathcal{S}^{(2)}_{\Delta,4}$. Here $\Delta: \ 0 = x_0 < x_1 < \ldots < x_n = 1$ is an arbitrary partition.

For $f \in C^2[0,1]$ and $z_i = \frac{x_i + x_{i+1}}{2}$, $i = 0, 1, \ldots, n-1$, they showed (see Theorem 1 in [9]) that there exists a unique $s_n \in \mathcal{S}^{(2)}_{\Delta,4}$ such that

(i) $s_n(z_i) = f(z_i), \;\; i = 0, 1, \ldots, n-1;$
(ii) $s''_n(x_i) = f''(x_i), \;\; i = 0, 1, \ldots, n;$
(iii) $s_n(0) = f(0), \;\; s_n(1) = f(1).$

This is one example for a situation in which the spline knots do not coincide with the interpolation nodes. The interpolation scheme is now as follows:

$$\begin{array}{llll}
x_0 & 1 & 0 & 1 \\
z_0 & 1 & 0 & 0 \\
x_1 & 0 & 0 & 1 \\
z_1 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
x_{n-1} & 0 & 0 & 1 \\
z_{n-1} & 1 & 0 & 0 \\
x_n & 1 & 0 & 1
\end{array}$$

Setting again $h := \max\{x_{\nu+1} - x_\nu : 0 \le \nu \le n-1\}$ and assuming that the "mesh ratios" satisfy

$$\beta_n \le K, \;\; n \in \mathbb{N},$$

they showed in Theorem 2 in [9] that

(i) for $f \in C^2[0,1]$, $\;\; \|(f - s_n)^{(k)}\| \le \varepsilon_{2,k} \cdot h^{2-k} \cdot \|f''\|, \;\; 0 \le k \le 2,$
(ii) for $f \in C^5[0,1]$, $\;\; \|(f - s_n)^{(k)}\| \le \varepsilon_{5,k} \cdot h^{5-k} \cdot \|f^{(5)}\|, \;\; 0 \le k \le 2.$

An application of Theorem 2.2 now leads to

Proposition 5.1. *Let s_n be the quartic Howell–Varma* $(0,2)$ *interpolant from above. Then, for all $f \in C^2[0,1]$,*

$$\|(f - s_n)^{(k)}\| \le c \cdot h^{2-k} \cdot \omega_3(f''; h), \;\; 0 \le k \le 2,$$

with c an absolute constant.

Proof. In Theorem 2.2 we set $k_u = 2$, $p_l = 2$, $p_u = 5$. This gives, putting $\delta = h$,

$$\begin{aligned}
\|(f - s_n)^{(k)}\| &\le c \cdot \left[h^{2-k} + h^{5-k} \cdot h^{-3}\right] \cdot \omega_3(f''; h) \\
&= c \cdot h^{2-k} \cdot \omega_3(f''; h), \;\; 0 \le k \le 2.
\end{aligned}$$

□

Corollary 5.2. *Proposition* 5.1 *implies all the separate statements in Howell's and Varma's Theorem* 2 *([9, p. 931]).*

REMARK 5.3. The authors mentioned also considered a different $(0,2)$ interpolation spline $q_n \in \mathcal{S}^{(2)}_{\Delta,4}$ satisfying conditions of the Fejér–type, namely

(i') $q_n(z_i) = f(z_i), \ \ i = 0,1,\dots,n-1;$
(ii') $q_n''(x_i) = 0, \ \ i = 0,1,\dots,n;$
(iii') $q_n(0) = f(0), \ \ q_n(1) = f(1).$

So the incidence matrix is the one from above, but, due to the Fejér–type conditions, the spline is now defined on all of $C[0,1]$. Howell and Varma (see [9, Theorem 3]) proved error estimates also in this case. It is beyond the scope of this note to also generalize and improve these.

6. BURKETT–VARMA INTERPOLANTS OF (0,1,2,4)–TYPE

J. Burkett and A. Varma [1] investigated $(0,1,2,4)$ spline interpolation in $\mathcal{S}^{(4)}_{\Delta,8}$. They proved that, for given $f \in C^4[0,1]$ and $z_i = \frac{x_i+x_{i+1}}{2}$, $i = 0,1,\dots,n-1$, there is a unique $s_n \in \mathcal{S}^{(4)}_{\Delta,8}$ satisfying

(i) $s_n^{(j)}(x_i) = f^{(j)}(x_i), \ \ i = 0,1,\dots,n; \ \ j = 0,1,2;$
(ii) $s_n^{(4)}(z_i) = f^{(4)}(z_i), \ \ i = 0,1,\dots,n-1;$
(iii) $s_n'''(x_0) = f'''(x_0), \ \ s_n'''(x_n) = f'''(x_n).$

Schematically, the interpolation requirements are again represented below:

$$
\begin{array}{lccccc}
x_0 & 1 & 1 & 1 & 1 & 0 \\
z_0 & 0 & 0 & 0 & 0 & 1 \\
x_1 & 1 & 1 & 1 & 0 & 0 \\
z_1 & 0 & 0 & 0 & 0 & 1 \\
x_2 & 1 & 1 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-1} & 1 & 1 & 1 & 0 & 0 \\
z_{n-1} & 0 & 0 & 0 & 0 & 1 \\
x_n & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Theorem 2 in [1] gives the following inequalities:

(i) for $f \in C^4E[0,1], \quad \|(f-s_n)^{(k)}\| \le \varepsilon_{4,k} \cdot h^{4-k} \cdot \|f^{(4)}\|, \ \ 0 \le k \le 4;$
(ii) for $f \in C^9[0,1], \quad \|(f-s_n)^{(k)}\| \le \varepsilon_{9,k} \cdot h^{9-k} \cdot \|f^{(9)}\|, \ \ 0 \le k \le 4.$

From them we derive

PROPOSITION 6.1. *Let* $s_n \in \mathcal{S}^{(4)}_{\Delta,8}$ *be the Burkett–Varma* $(0,1,2,4)$ *interpolant. Then, for all* $f \in C^4[0,1]$,

$$\|(f-s_n)^{(k)}\| \le c \cdot h^{4-k} \cdot \omega_5(f^{(4)};h), \quad 0 \le k \le 4.$$

Proof. We now use $k_u = 4, p_l = 4, p_u = 9$ in Theorem 2.2 and put $\delta_n(p,k) = h^{p-k}$ for $p \in \{4,9\}, \delta = h$. This gives

$$\begin{aligned}\|(f-s_n)^{(k)}\| &\le c \cdot \left[h^{4-k} + h^{9-k} \cdot h^{-5}\right] \cdot \omega_5(f^{(4)};h)\\ &\le c \cdot h^{4-k} \cdot \omega_5(f^{(4)};h),\end{aligned}$$

which was our claim. □

A different type of $(0,1,2,4)$ interpolation problem was investigated by the same authors in [2]. They showed existence and unicity of a spline $q_n \in \mathcal{S}^{(4)}_{\Delta,8}$ such that, for $f \in C^4[0,1]$,

(i) $q_n(z_i) = f(z_i), \quad i = 0,1,\ldots,n-1;$
(ii) $q_n^{(j)}(x_i) = f^{(j)}(x_i), \quad i = 0,1,\ldots,n; \quad j = 1,2,4;$
(iii) $q_n(x_0) = f(x_0), \quad q_n(x_n) = f(x_n).$

Thus the "incidence matrix" now attains the form

$$\begin{array}{llllll}
x_0 & 1 & 1 & 1 & 0 & 1\\
z_0 & 1 & 0 & 0 & 0 & 0\\
x_1 & 0 & 1 & 1 & 0 & 1\\
z_1 & 1 & 0 & 0 & 0 & 0\\
x_2 & 0 & 1 & 1 & 0 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
x_{n-1} & 0 & 1 & 1 & 0 & 1\\
z_{n-1} & 1 & 0 & 0 & 0 & 0\\
x_n & 1 & 1 & 1 & 0 & 1
\end{array}$$

It was shown in Theorem 2 of [2] that, if the mesh ratios β_n remain bounded, the following are valid:

(i) for $f \in C^4[0,1]$, $\quad \|(f-q_n)^{(k)}\| \le \varepsilon_{4,k} \cdot h^{4-k} \cdot \|f^{(4)}\|, \ \ 0 \le k \le 4;$
(ii) for $f \in C^9[0,1]$, $\quad \|(f-q_n)^{(k)}\| \le \varepsilon_{9,k} \cdot h^{9-k} \cdot \|f^{(9)}\|, \ \ 0 \le k \le 4.$

In Theorem 2.2 we put again $k_u = 4, p_l = 4, p_u = 9$ and arrive at

PROPOSITION 6.2. *Let q_n be the Burkett–Varma spline interpolant as described above. Then, for all $f \in C^4[0,1]$,*

$$\|(f-q_n)^{(k)}\| \le c \cdot h^{4-k} \cdot \omega_5(f^{(4)};h), \quad 0 \le k \le 4.$$

7. BURKETT–VARMA INTERPOLANTS–(0,1,3) CASE

This case was also considered in [2]. The authors showed that for $f \in C^3E[0,1]$ there exists a unique $s_n \in \mathcal{S}^{(3)}_{\Delta,7}$ such that

(i) $s_n^{(j)}(x_i) = f^{(j)}(x_i), \quad i = 0,1\ldots,n; \quad j = 0,3;$
(ii) $s_n^{(j)}(z_i) = f^{(j)}(z_i), \quad i = 0,1,\ldots,n-1; \quad j = 0,1;$
(iii) $s_n'(x_0) = f'(x_0), \quad s_n'(x_n) = f'(x_n).$

The visualization is now:

$$\begin{array}{lcccc}
x_0 & 1 & 1 & 0 & 1 \\
z_0 & 1 & 1 & 0 & 0 \\
x_1 & 1 & 0 & 0 & 1 \\
z_1 & 1 & 1 & 0 & 0 \\
x_2 & 1 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-1} & 1 & 0 & 0 & 1 \\
z_{n-1} & 1 & 1 & 0 & 0 \\
x_n & 1 & 1 & 0 & 1
\end{array}$$

From Theorem 4 in [2] we derive the following: If the mesh ratios β_n remain bounded, then

(i) for $f \in C^3[0,1], \quad \|(f-s_n)^{(k)}\| \le \varepsilon_{3,k} \cdot h^{3-k} \cdot \|f'''\|, \quad 0 \le k \le 3;$
(ii) for $f \in C^8[0,1], \quad \|(f-s_n)^{(k)}\| \le \varepsilon_{8,k} \cdot h^{8-k} \cdot \|f^{(8)}\|, \quad 0 \le k \le 3.$

Using Theorem 2.2 now with $k_u = 3, p_l = 3, p_u = 8$ gives

PROPOSITION 7.1. *Let $s_n \in \mathcal{S}^{(3)}_{\Delta,7}$ be the Burkett–Varma $(0,1,3)$ spline interpolant. Then for all $f \in C^3[0,1]$ we have*

$$\|(f-s_n)^{(k)}\| \le c \cdot h^{3-k} \cdot \omega_5(f''';h), \quad 0 \le k \le 3.$$

8. CONCLUSION

(i) In the above we have restricted ourselves to the consideration of the now classical quintic $(0,2)$ spline interpolant of Meir and Sharma, to the generalization due to Demko, and to that of certain further lacunary spline operators which were discussed more recently by Varma and his collaborators. Inequalities similar to the ones given above can also be derived for cases which were investigated in further papers such as [19], [20], [14], and others.

(ii) As was already pointed out in the book by Lorentz, Jetter and Riemenschneider (see [12, p. 190]), "interpolation by spline functions is a more complex subject than polynomial interpolation". This is evident in particular when the many possible choices of interpolation matrices E (using interpolation <u>nodes</u>) meet the many possible choices of Schoenberg spaces $\mathcal{S}_{\Delta,q}^{(r)}$ (using spline <u>knots</u>).

Not even the fundamental problems of existence, uniqueness and representation seem to have been treated to a satisfactory extent until the time of this writing. The present note should be considered as a possible guideline for future research concerning the quantitative aspect of the matter.

REFERENCES

[1] Burkett, J. and Varma, A. K., *Lacunary interpolation by splines,* $(0,1,2,4)$ *case*, in: Constructive Theory of Functions (Proc. Int. Conf. Varna 1991; ed. by K.G. Ivanov et al.), pp. 9–17, Publ. House of the Bulgarian Academy of Sciences, Sofia, 1992.

[2] Burkett, J. and Varma, A. K., *Lacunary spline interpolation*, Suppl. Rend. Circ. Mat. Palermo, (2) **33**, pp. 219–228, 1993.

[3] Carlson, R. E. and Hall, C. A., *Bicubic spline interpolation in rectangular polygons*, J. Approx. Theory, **6**, pp. 366–377, 1972.

[4] Demko, S., *Lacunary polynomial spline interpolation*, SIAM J. Numer. Anal., **13**, pp. 369–391, 1976.

[5] DeVore, R. A. and Lorentz, G.G., *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

[6] Gonska, H. H., *Degree of approximation by lacunary interpolators:* $(0,\ldots,R-2,R)$ *interpolation*, Rocky Mountain J. Math., **19**, pp. 157–171, 1989.

[7] Gonska, H. H., *Degree of simultaneous approximation of bivariate functions by Gordon operators*, J. Approx. Theory, **62**, pp. 170–191, 1990.

[8] Gonska, H. H. and Knoop, H. B., *On Hermite–Fejér Interpolation: A Bibliography* (1914–1987), Studia Sci. Math. Hungar., **25**, pp. 147–198, 1990.

[9] Howell, G. and Varma, A. K., $(0,2)$ *interpolation with quartic splines*, Numer. Funct. Anal. Optimization, **11**, pp. 929–936, 1991.

[10] Kacsó, D. P. and Gonska, H. H., *Simultaneous approximation by* $(0,\ldots,R-2,R)$ *interpolators*, Calcolo, **32**, pp. 177–191, 1995.

[11] Kacsó, D. P. and Gonska, H. H., *Simultaneous approximation by Birkhoff interpolators*, Rocky Mountain J. Math., **26**, pp. 1303–1320, 1998.

[12] Lorentz, G. G., Jetter, K. and Riemenschneider, S. D., *Birkhoff Interpolation*, Addison–Wesley, Reading, MA, 1983.

[13] Meir, A. and Sharma, A., *Lacunary interpolation by splines*, SIAM J. Numer. Anal., **10**, pp. 433–442, 1973.

[14] Prasad, J. and Varma, A. K., *Lacunary interpolation by quintic splines*, SIAM J. Numer. Anal., **16**, pp. 1075–1079, 1979.

[15] Sharma, A. and Meir, A., *Degree of approximation of spline interpolation*, J. Math. Mech., **15**, pp. 759–767, 1966.

[16] Schoenberg, I. J., *On the Ahlberg–Nilson extension of spline interpolation: The g-splines and their optimal properties*, J. Math. Anal. Appl., **21**, pp. 207–231, 1968.

[17] Schumaker, L. L., *Spline Functions: Basic Theory*, John Wiley and Sons, New York, 1981.

[18] Swartz, B. and Varga, R. S., *A note on lacunary interpolation by splines*, SIAM J. Numer. Anal., **10**, pp. 443–447, 1973.

[19] Varma, A. K., *Lacunary interpolation by splines, I*, Acta Math. Acad. Sci. Hung., **31**, pp. 185–192, 1978.

[20] Varma, A. K., *Lacunary interpolation by splines, II*, Acta Math. Acad. Sci. Hung., **31**, pp. 193–203, 1978.

Received December 12, 2000.