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THE CREST FACTOR FOR TRIGONOMETRIC POLYNOMIALS PART I: APPROXIMATION THEORETICAL ESTIMATES

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Dedicated to Professor Werner Haußmann on his sixtieth birthday.

Abstract. The Chebyshev norm of a degree n trigonometric polynomial is estimated against a discrete maximum norm based on equidistant sampling points where, typically, oversampling rather than critical sampling is used. The bounds are derived from various methods known from classical Approximation Theory. These estimates are of fundamental importance for the design of efficient OFDM in communication systems.

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1. INTRODUCTION

In modern digital communication a channel input signal is generally synthesized as a linear combination of certain bases functions whose coefficients are bearing the information that is to be transmitted [3], [10]. In the popular orthogonal frequency division multiplexing (OFDM) communications system the signal is expanded in terms of an orthogonal trigonometric basis and, up to a modulation factor $e^{i\xi t}$, it has the form

$$p(t) = a_0 + \sum_{k=1}^{n} \Big(a_k \cos(kt) + b_k \sin(kt) \Big). \tag{1.1}$$

Here, the degree n represents the width of the base band, which is generally restricted.

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One of the problems we may encounter in OFDM based transmission is a large peak-to-average ratio (PAR), or equivalently, a large crest factor (CF) for signals, given by

$$\text{CF} = \sqrt{\text{PAR}} := \frac{\|p\|_\infty}{\|p\|_2} = \frac{\max_{t\in[0,2\pi]} |p(t)|}{\left(\frac{1}{2\pi}\int_0^{2\pi} |p(t)|^2\,dt\right)^{1/2}}. \tag{1.2}$$

The reliability of transmission systems containing input signals with large CF is reduced. This is due to the fact that power amplifiers are only capable of modulating signals which are bounded by a fixed constant, the so-called clip level. Any input signal exceeding this value is clipped at this ("cut-off") level. This introduces noise to the system, reduces the signal-to-noise ratio (SNR) and thus has strong impact on the reliability of the system. In order to weaken this effect one can reduce the amplitude of the signal. But this worsens the SNR directly. The total SNR of the system is also reduced due to the increased quantization noise. The trade-off between clipping and reducing the amplitude is chosen such that the maximal possible SNR of the transmission system is achieved.

There exists a variety of methods aiming at a reduction of the CF in an OFDM based transmission system; see [1], [2], [6], [12]. Many of them require a fast and precise estimate of the CF of a given input signal, where the estimate should be based on readily available information, such as the coefficients in (1.1) and the sampled values of the signal on a sufficiently dense equidistant mesh. Clearly, the denominator of (1.2) can be expressed in terms of the coefficients of the signal (1.1), via Parseval's identity. This gives rise to the central problem discussed in this paper, which is equivalent to estimating the crest factor: We want to derive bounds of the Chebyshev norm

$$\|p\|_\infty := \max_{t\in[0,2\pi]} |p(t)|$$

of a trigonometric polynomial p of degree at most n, i.e.

$$p \in T_n := \text{span}\ \{1, \sin t, \cos t, \ldots, \sin nt, \cos nt\},$$

in terms of a discrete maximum norm

$$\|p\|_{N,\infty} := \max_{k=0,\ldots,N-1} |p(t_k)|, \tag{1.3}$$

where the t_k are the N equidistant sampling points

$$\Theta_N := \left\{ t_k = t_k^{(N)} = k\frac{2\pi}{N} \;\middle|\; k = 0, \ldots, N-1 \right\}. \tag{1.4}$$

In other words:

For given natural numbers n (the maximal degree of the trigonometric polynomials) and N (the number of equidistant sampling points), we are after the optimal constants $c_{n,N} > 0$ in the estimate

$$\|p\|_{N,\infty} \le \|p\|_\infty \le c_{n,N}\|p\|_{N,\infty} \qquad \text{for all } p \in T_n. \tag{1.5}$$

Obviously, such a constant can not exist if $N < 2n+1$, since then T_n contains a non-zero polynomial vanishing at all sampling points. Therefore, we shall always assume that

$$N \ge 2n+1.$$

For the case of critical sampling, i.e., $N = 2n+1$, estimates for $c_{n,N}$ have been known for quite a while. In this case, the optimal constant in (1.5) is given by the so-called Lebesgue constant for interpolation with trigonometric polynomials; see [5], [11], where the case of algebraic polynomials was discussed also. If the continuous and discrete Chebyshev norms are replaced by p-norms, then the corresponding estimates are named after Marcinkiewicz and Zygmund; see [13, Ch.X.7], or [9] for a more recent paper.

For the case of oversampling, the only results known to us are the estimate

$$c_{n,2m} \le \frac{1}{\cos\frac{\pi n}{2m}}$$

due to Ehlich and Zeller [4], which is sharp if and only if $n|m$, and the recent result by Wunder and Boche [12],

$$c_{n,N} \le \sqrt{\tfrac{N+\ 2n+1}{N-(2n+1)}}.$$

We embark on this important problem by elaborating on basic methods and results from classical Approximation Theory. Three different aspects will be considered. In Section 2, we describe a straightforward application of Bernstein's inequality in order to get a rough and easy estimate (Proposition 1). Finer estimates are derived through the use of summation kernels in Section 3, where we show that

$$c_{n,N} \le \sqrt{\tfrac{N}{N-2n}}$$

(Theorem 1). In Section 4, we develop a characterization of (n, N)-extremal polynomials—for which the right-hand side estimate in (1.5) is sharp—in terms of alternation properties (Theorem 2). These properties of extremal polynomials enable us to show that $c_{n,N}$ depends on $\frac{N}{n}$ only, and they provide a powerful tool to ex-actly determine the constants $c_{n,N}$ in special cases. Some of these examples and some comments on surprising properties of the function $\frac{N}{n} \mapsto c_{n,N}$ conclude the paper.

2. OVERSAMPLING AND BERNSTEIN'S INEQUALITY

We want to sample trigonometric polynomials $p \in T_n$ of degree at most n at N equidistant points within one period, i.e., on the sampling set (1.4), and then recover the polynomials from this information.

Since T_n is $2n+1$-dimensional, the minimal setup is given by $N = 2n+1$. In this case, sampling and recovering is usually denoted *trigonometric interpolation* involving an odd number of equidistant points. Here,

$$\begin{aligned}\Gamma_k(t) &:= \frac{1}{2n+1}\,\frac{\sin\left(\frac{2n+1}{2}\right)(t-t_k)}{\sin\frac{1}{2}(t-t_k)}\\ &= \frac{1}{2n+1}\left(1+2\sum_{j=1}^{n}\cos j(t-t_k)\right), \quad k=0,1,\ldots,2n,\end{aligned}$$

are the so-called fundamental polynomials in T_n interpolating the δ-data, i.e., $\Gamma_k(t_\ell) = \delta_{k,\ell}$ for $k, \ell = 0, 1, \ldots, 2n$. Consequently, polynomials $p \in T_n$ can be recovered from the information $p(t_k)$, $k = 0, 1, \ldots, 2n$, by the Lagrange interpolation formula

$$p = \sum_{k=0}^{2n} p(t_k)\,\Gamma_k \qquad \text{for all } p \in T_n. \tag{2.1}$$

For this case of critical sampling we have

$$c_{n,2n+1} = \lambda_n,$$

where $\lambda_n := \|\Lambda_n\|_\infty$ is the Chebyshev norm of the so-called Lebesgue function

$$\Lambda_n(t) := \sum_{k=0}^{2n} |\Gamma_k(t)|.$$

It is well-known (see, e.g., [13, Ch.X]) that

$$\lambda_n = \tfrac{2}{\pi}\log n + \mathcal{O}(1) \qquad (n \to \infty),$$

and for this reason, trigonometric interpolation or critical sampling of polynomials with high degree is often called "unstable".

In order to avoid this instability, we can use a higher sampling rate. Let us denote

$$q := \frac{N}{n}. \tag{2.2}$$

We will show later that $c_{n,N}$ actually depends on q only. Let us begin with a first estimate.

PROPOSITION 1. *For $q = \frac{N}{n} > \pi$, we have*

$$c_{n,N} \le \frac{q}{q-\pi}.$$

Proof. The result is an immediate consequence of the classical Bernstein inequality $\|p'\|_\infty \le n\|p\|_\infty$ for $p \in T_n$ (see [8, Ch.3.2]), if we use the integrated form

$$|p(t) - p(t')| \le n\,|t - t'|\,\|p\|_\infty, \quad p \in T_n.$$

Without loss of generality we may assume $\|p\|_\infty = 1$. Consider $t^* \in [0, 2\pi[$ with $|p(t^*)| = 1$, and choose $t_k \in \Theta_N$ such that t_k is closest to t^*. Then $|t_k - t^*| \le \frac{\pi}{N}$ and thus

$$|p(t_k) - 1| \le n\,|t_k - t^*| \le \frac{n\pi}{N} = \frac{\pi}{q}.$$

Hence,

$$\|p\|_{N,\infty} \ge |p(t_k)| \ge \left(1 - \tfrac{\pi}{q}\right)\|p\|_\infty.$$

□

REMARK. Theorem 1 below will give a sharper estimate. However, we find it interesting enough to include this proposition here, since it is based on a simple argument using a Bernstein-Markov type inequality, and thus the method of proof applies in various other situations as well. *E.g.*, in [7], this idea has been elaborated on in order to construct so-called norming sets for subspaces of continuous functions on compact manifolds by using oversampling as well. □

3. SUMMATION KERNELS

The Lagrange type interpolation formula (1.1) can be written in terms of the n-th degree *Dirichlet kernel*

$$D_n(t) := \sum_{k=-n}^{+n} e^{ikt} = 1 + 2\sum_{k=1}^{n} \cos kt = \frac{\sin\frac{2n+1}{2}t}{\sin\frac{1}{2}t},$$

which is the kernel of the n-th degree Fourier projection operator

$$S_n \,:\, C_{2\pi} \to T_n, \quad (S_n f)(t) = \frac{1}{2\pi}\int_0^{2\pi} D_n(t-x)\, f(x)\, dx.$$

Namely, $\Gamma_k = \dfrac{1}{2n+1} D_n(\cdot - t_k)$, $k = 0, \dots, 2n$. Using "gliding averages" of such operators (respectively, their Dirichlet kernels), we can reproduce formula (2.1) with modified functions Γ_k. In this way we shall improve on Proposition 1.

A family of such operators is the class of *(generalized) de la Vallée–Poussin means*

$$S_{n,m} := \frac{1}{m-n}\big(S_n + S_{n+1} + \cdots + S_{m-1}\big) \qquad \text{for } m > n \tag{3.1}$$

with corresponding summation kernels

$$D_{n,m}(t) = \frac{1}{m-n}\,\frac{\sin\frac{m+n}{2}t\,\sin\frac{m-n}{2}t}{\sin^2\frac{1}{2}t}. \tag{3.2}$$

As special cases, we obtain the *Fejér operators* $S_{0,n}$ with corresponding Fejér kernels

$$F_n(t) := D_{0,n}(t) = \frac{1}{n}\,\frac{\sin^2 n\frac{t}{2}}{\sin^2\frac{t}{2}},$$

and also the Fourier projectors themselves as $S_n = S_{n,n+1}$. By construction,

$$S_{n,m} \,:\, C_{2\pi} \to T_{m-1}, \quad (S_{n,m} f)(t) = \frac{1}{2\pi}\int_0^{2\pi} D_{n,m}(t-x)\, f(x)\, dx, \tag{3.3}$$

and

$$S_{n,m}\, p = p \qquad \text{for all } p \in T_n.$$

In particular, choosing $p \equiv 1$ yields

$$\frac{1}{2\pi}\int_0^{2\pi} D_{n,m}(t-x)\, dx = 1.$$

The operator norm of these operators is given by

$$\|S_{n,m}\| = \frac{1}{2\pi}\int_0^{2\pi} |D_{n,m}(t)|\,dt = \frac{4}{\pi^2}\log\frac{m}{m-n} + \mathcal{O}(1) \tag{3.4}$$

(see [8, p. 110]), hence these norms are uniformly bounded in n if m is chosen proportional to n.

Lemma 1. *Given the sampling set* $\Theta_N = \{t_k = k\frac{2\pi}{N} | k = 0, \ldots, N-1\}$, *the de la Vallée–Poussin summation kernels* $D_{n,m}$ *have the property*

$$p(t) = \frac{1}{N}\sum_{k=0}^{N-1} p(t_k)D_{n,m}(t - t_k) \qquad \text{for all } p \in T_n$$

whenever $N \geq m+n$.

Proof. Given $p \in T_n$, we have

$$p(t) = \big(S_{n,m}\,p\big)(t) = \frac{1}{2\pi}\int_0^{2\pi} D_{n,m}(t-x)\,p(x)\,dx.$$

As a function of the variable x, the integrand is a trigonometric polynomial of degree at most $n+m-1$. Thus the integral can be evaluated exactly by applying a rectangular rule with sufficiently many points. Based on the set of knots Θ_N, the rectangular rule will do this whenever $N-1 \geq n+m-1$, which yields our claim. □

Theorem 1. *Given* $m \geq n+1$ *and* $N \geq m+n$, *we have*

$$\|p\|_\infty \leq \sqrt{\frac{m+n}{m-n}}\,\|p\|_{N,\infty} \qquad \text{for all } p \in T_n.$$

In particular,

$$c_{n,N} \leq \sqrt{\frac{N}{N-2n}} \qquad \text{for } N \geq 2n+1.$$

Our proof is based on ideas from [12], where it has been shown that

$$c_{n,N} \leq \sqrt{\tfrac{N+\ 2n+1}{N-(2n+1)}}.$$

Using properties of summation kernels, we can both shorten their argument and somewhat improve their result.

Proof. The second estimate follows from the first one by choosing $m := N - n$. Thus, in view of Lemma 1, it suffices to show that

$$\frac{1}{N}\sum_{k=0}^{N-1}\left|D_{n,m}(t-t_k)\right| \le \sqrt{\frac{m+n}{m-n}}. \tag{3.5}$$

According to (3.2), the left-hand side is given by

$$\frac{1}{N(m-n)}\sum_{k=0}^{N-1}\Big|\frac{\sin\frac{m+n}{2}(t-t_k)\,\sin\frac{m-n}{2}(t-t_k)}{\sin^2\frac{1}{2}(t-t_k)}\Big|.$$

Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big(\sum_{k=0}^{N-1}\Big|\frac{\sin\frac{m+n}{2}(t-t_k)\,\sin\frac{m-n}{2}(t-t_k)}{\sin^2\frac{1}{2}(t-t_k)}\Big|\Big)^2\\
&\quad\le \sum_{k=0}^{N-1}\frac{\sin^2\frac{m+n}{2}(t-t_k)}{\sin^2\frac{1}{2}(t-t_k)}\sum_{k=0}^{N-1}\frac{\sin^2\frac{m-n}{2}(t-t_k)}{\sin^2\frac{1}{2}(t-t_k)}\\
&\quad= (m+n)\Big(\sum_{k=0}^{N-1}D_{0,m+n}(t-t_k)\Big)(m-n)\Big(\sum_{k=0}^{N-1}D_{0,m-n}(t-t_k)\Big)\\
&\quad= (m+n)\,N\,(m-n)\,N,
\end{aligned}$$

where in the last step, we applied Lemma 1 to the constant function $p(t)\equiv 1$. This estimate yields (3.5). $\square$

In engineering, one often prefers to use the so-called oversampling rate $r = \frac{N}{2n+1}$. In terms of this quantity, we obtain the following results. Here, $\lfloor\cdot\rfloor$ as usual denotes the largest integer function.

Corollary 1. (i) *Let* $N > 2n+1$ *and* $r := \frac{N}{2n+1}$. *Then*

$$c_{n,N} \le \sqrt{\frac{r}{r-1}}.$$

(ii) *Fix* $r > 1$, *and let* $N(n) := \lfloor r(2n+1)\rfloor$. *Then*

$$c_{n,N(n)} \le \sqrt{\frac{r}{r-1}} \quad \textit{for all } n\in\mathbb{N}.$$

Proof. (i) is obvious, and (ii) follows from

$$\frac{\lfloor r(2n+1)\rfloor}{\lfloor r(2n+1)\rfloor - 2n} \le \frac{r(2n+1)}{r(2n+1)-1-2n} = \frac{r}{r-1}.$$

$\square$

4. EXTREMAL POLYNOMIALS AND EXTREMAL ALTERNATING SETS

The optimal constant $c_{n,N}$ in estimate (1.5) can be expressed as

$$c_{n,N} = \sup_{p \in T_n \setminus \{0\}} \frac{\|p\|_\infty}{\|p\|_{N,\infty}}. \tag{4.1}$$

A polynomial $p^* \in T_n$ will be called (n,N)-*extremal*, if it satisfies

$$c_{n,N} = \frac{\|p^*\|_\infty}{\|p^*\|_{N,\infty}}.$$

Such an extremal polynomial always exists whenever $N \geq 2n{+}1$. This follows by a standard compactness argument: Given the unit ball in T_n, i.e.

$$B_n := \{p \in T_n \mid \|p\|_\infty = 1\},$$

we may write

$$c_{n,N} = \sup_{p \in B_n} \frac{1}{\|p\|_{N,\infty}}.$$

For $N \geq 2n{+}1$, the mapping

$$B_n \to \mathbb{R}, \quad p \mapsto \|p\|_{N,\infty}$$

is nonzero and continuous. Since B_n is compact, this function assumes its minimum at some p^*.

For $n = 0$, problem (4.1) is trivial, since any constant polynomial $p \in T_0 \setminus \{0\}$ is $(0,N)$-extremal, and $c_{0,N} = 1$ for any N. Therefore, we shall from now on tacitly assume

$$n \geq 1.$$

Consequently, we have for all $N \geq 2n{+}1$ that

$$c_{n,N} > 1, \tag{4.2}$$

which can be seen by simply considering an appropriate element of T_n. For example, the polynomial

$$p_N(x) = \cos(x - \tfrac{\pi}{N}) + 1$$

satisfies $p_N \in T_1$ and $c_{n,N} \geq \|p_N\|_\infty / \|p_N\|_{N,\infty} > 1$.

We shall characterize extremal polynomials by an alternation type theorem. To this end, given $p \in T_n \setminus T_0$, let

$$E_p := \{t \in [0, 2\pi[\;|\; |p(t)| = \|p\|_\infty\}.$$

Since p is continuous, E_p is not empty. On the other hand, it contains at most $2n$ points, since p attains a local extremum at each $t \in E_p$, and p' has at most $2n$ zeros in $[0, 2\pi[$. From (4.2), we know that if p is (n, N)-extremal, then

$$E_p \cap \Theta_N = \emptyset.$$

So we may choose $t^* \in E_p$ and, after replacing p by $p(\cdot - t_k)$ with an appropriate k if necessary, assume that $0 < t^* < t_1$.

Furthermore, let

$$E_{N,p} := \{t_k \in \Theta_N : |p(t_k)| = \|p\|_{N,\infty}\}$$

and define

$$E^*_{N,p} := E_{N,p} \cup \{t^*, t^* + 2\pi\}.$$

Now we group the elements

$$t^* < t_{j_1} < t_{j_2} < \cdots < t_{j_m} < t^* + 2\pi$$

of $E^*_{N,p}$ into adjacent groups

$$\Delta_1 < \Delta_2 < \cdots < \Delta_{2\kappa+1} \tag{4.3}$$

according to the sign of the function values

$$-p(t^*),\ p(t_{j_1}),\ p(t_{j_2}),\ \ldots,\ p(t_{j_m}),\ -p(t^* + 2\pi) \tag{4.4}$$

in the sense that in each Δ_ℓ, the sign stays constant, and in consecutive groups, the sign alternates. (The reason for taking $p(t^*)$ with the opposite sign in (4.4) will become clear in the proof of Theorem 2.)

Because of the periodicity of p, it is clear that there must be an odd number of groups in (4.3), which we indicated by the index $2\kappa+1$, and it is natural to count

$$\Delta_1 \cup \Delta_{2\kappa+1} =: \Delta_1^* \tag{4.5}$$

as one group. Choosing an element t_{k_ℓ} from each Δ_ℓ for $\ell = 2, 3, \ldots, 2\kappa$, we obtain a set of points

$$t^* < t_{k_2} < t_{k_3} < \cdots < t_{k_{2\kappa}} < t^* + 2\pi \tag{4.6}$$

with the property that the sequence

$$(4.7) \qquad -p(t^*),\, p(t_{k_2}),\, p(t_{k_3}),\, \ldots,\, p(t_{k_{2\kappa}}),\, -p(t^*{+}2\pi) = -p(t^*)$$

is strictly alternating in sign; in particular,

$$|-p(t^*)| = \|p\|_\infty \quad \text{and} \quad p(t_{k_\ell}) = (-1)^\ell \operatorname{sign}(p(t^*))\, \|p\|_{N,\infty}.$$

Such a set (4.6) will be called an *N-extremal alternating set of length 2κ for p*. It is defined for any p with the property $\|p\|_\infty > \|p\|_{N,\infty}$.
REMARK. Let us again point to the fact that in contrast to the situation in Chebyshev's alternation theorem for uniform approximation with Chebyshev systems, the point t^* is counted in an exceptional way: We consider $-p(t^*)$ rather than $p(t^*)$ in (4.7), and in addition $|p(t^*)| > \|p\|_{N,\infty}$. Also note that since we are dealing with trigonometric polynomials of degree n, the upper estimate

$$2\kappa \le 2n{+}2$$

for the number of groups is immediate.

With these preparations, we can state a necessary and sufficient condition for a trigonometric polynomial to be extremal.

THEOREM 2. *Let $n > 0$ and $N \ge 2n{+}1$. For $p^* \in T_n$, the following are equivalent:*

(a) *p^* is (n,N)-extremal,*
(b) *$\|p^*\|_\infty > \|p^*\|_{N,\infty}$, and p^* has an N-extremal alternating set of length $2n{+}2$.*

Furthermore, if p^* is (n,N)-extremal, it has the following properties:

(i) *$p^* \in T_n \setminus T_{n-1}$.*
(ii) *p^* is unique up to translation by $k\frac{2\pi}{N}$ and multiplication by a constant.*
(iii) *If t^* satisfies $|p^*(t^*)| = \|p^*\|_\infty$, then t^* is of the form $\frac{(2k+1)\pi}{N}$, and p^* has even symmetry about t^*.*

Proof. (a) $\Rightarrow$ (b): Assume that p^* is (n,N)-extremal. By (4.2), this implies $\|p^*\|_\infty > \|p^*\|_{N,\infty}$. Construct an N-extremal alternating set for p^*. If its length satisfies $2\kappa \le 2n$, we can choose points $\tau_1, \ldots, \tau_{2\kappa} \in [0, 2\pi[$ such that

$$\Delta_1 < \tau_1 < \Delta_2 < \tau_2 < \cdots < \Delta_{2\kappa} < \tau_{2\kappa} < \Delta_{2\kappa+1}$$

and a function $q \in T_\kappa \subseteq T_n$ such that q has a zero of order 1 at each τ_ℓ, $\ell = 1, \ldots, 2\kappa$, and no other zeros in the interval $[0, 2\pi[$. We can also assume that $\|q\|_\infty = 1$, andthat q is mimicking the sign distribution of p^* in the sense that

$$p^*(t)\,q(t)\begin{cases}<0 \text{ for } t\in\Delta_1^*\\ >0 \text{ for } t\in\Delta_\ell,\ \ell=2,\dots,2\kappa.\end{cases}$$

Letting $\alpha = \|p^*\|_{N,\infty} - \max\{|p^*(t_k)|\,|\,t_k \notin E_{N,p}\} > 0$, we obtain with $\widetilde{p} := p^* - \alpha\, q$ an element of T_n satisfying

$$\|\widetilde{p}\|_\infty \ge |\widetilde{p}(t^*)| > |p^*(t^*)| = \|p^*\|_\infty \quad\text{and}\quad \|\widetilde{p}\|_{N,\infty} \le \|p^*\|_{N,\infty},$$

in contradiction to p^* being extremal. So we may conclude that $2\kappa = 2n+2$.
(b) $\Rightarrow$ (a): Assume that p^* satisfies $\|p^*\|_\infty > \|p^*\|_{N,\infty}$ and has an N-extremal alternating set of length $2n+2$. If necessary, shift p^* by $k\frac{2\pi}{N}$ and multiply by a constant to ensure that the t^* in the N-extremal alternating set satisfies $0 < t^* < \frac{2\pi}{N}$ and $p^*(t^*) = \|p^*\|_\infty$, and that $\|p^*\|_{N,\infty} = 1$.

Now let $\widetilde{p}$ be (n,N)-extremal. Again, after shifting and normalizing $\widetilde{p}$, we may assume that $\widetilde{p}(\widetilde{t}) = \|\widetilde{p}\|_\infty$ for some $\widetilde{t} \in\,]0, \frac{2\pi}{N}[$, and that $\|\widetilde{p}\|_{N,\infty} = 1$.

Then $\widetilde{p} - p^*$ satisfies

$$(\widetilde{p} - p^*)(\widetilde{t}) = \|\widetilde{p}\|_\infty - p^*(\widetilde{t}) \ge \|\widetilde{p}\|_\infty - \|p^*\|_\infty \ge 0$$

by the extremality of $\widetilde{p}$, and

$$(-1)^{\ell-1}\,(\widetilde{p} - p^*)(t_{k_\ell}) = (-1)^{\ell-1}\,\widetilde{p}(t_{k_\ell}) + \|p^*\|_{N,\infty} \ge 0, \qquad \ell = 2,3,\dots,2n{+}2$$

at the points t_{k_ℓ} of the N-extremal alternating set of p^*. But since $\widetilde{p} - p^* \in T_n$, this sign distribution implies that $\widetilde{p} - p^* \equiv 0$.

In particular, this implies the stated uniqueness property (ii) of p^*. Also, the number of sign changes requires (i) $p^* \in T_n \setminus T_{n-1}$.

It remains to show the symmetry property (iii). To this end, assume that p^* is (n,N)-extremal with $|p^*(t^*)| = \|p^*\|_\infty$. Again, we may assume after a translation by $k\frac{2\pi}{N}$ that $0 < t^* < \frac{2\pi}{N}$. Let $\widetilde{p}(t) = p^*(\frac{2\pi}{N} - t)$ and $\widetilde{t} = \frac{2\pi}{N} - t^*$. Then $\widetilde{p}$ is also (n,N)-extremal with $|\widetilde{p}(\widetilde{t})| = \|\widetilde{p}\|_\infty$, and by the same reasoning as above in showing (b) $\Rightarrow$ (a), we conclude that $\widetilde{p} = p^*$. This implies $|p^*(\widetilde{t})| = |p^*(t^*)| = \|p^*\|_\infty$. Bycounting the zeros of $(p^*)'$, we see that p^* can have only one extremum in the interval $[0, \frac{2\pi}{N}]$, so we necessarily have $\widetilde{t} = t^* = \frac{\pi}{N}$. □

The line of argument in this proof illustrates the exceptional role of the sign of p at the point $t = t^*$ in (4.4): To maximize the quotient $\|p\|_\infty / \|p\|_{N,\infty}$, we want to keep $|p(t_k)|$ small for each k, and at the same time increase $\|p\|_\infty = |p(t^*)|$.

To illustrate the properties of an (n, N)-extremal function, Figure 1 shows a typical example p^* for $(n, N) = (3, 11)$ and indicates the N-extremal alternating set.

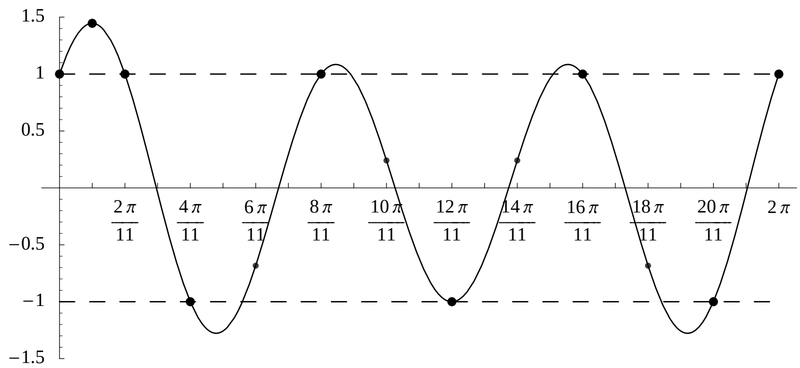

Fig. 1. $p^*_{3,11}$ with its 11-extremal alternating set.

The characterization of extremal polynomials in Theorem 2 has a number of immediate consequences.

Corollary 2. *Let $n > 0$ and $N \geq 2n + 1$. Then*

$$c_{n,N} = c_{kn,kN} \qquad \text{for all } k \in \mathbb{N}.$$

Moreover, if $p^*_{n,N}$ is (n, N)-extremal, then

$$p^*_{kn,kN} := p^*_{n,N}(k\,\cdot\,)$$

is (kn, kN)-extremal.

Proof. Since $p^*_{n,N} \in T_n$ satisfies $\|p^*_{n,N}\|_\infty > \|p^*_{n,N}\|_{N,\infty}$ and has an N-extremal alternating set of length $2n+2$, we find that $p^*_{n,N}(k\,\cdot\,) \in T_{kn}$ satisfies

$$\|p^*_{n,N}(k\,\cdot\,)\|_\infty > \|p^*_{n,N}(k\,\cdot\,)\|_{kN,\infty}$$

and has a kN-extremal alternating set of length $2kn+2$. Consequently,

$$c_{kn,kN} = \frac{\|p^*_{n,N}(k\,\cdot\,)\|_\infty}{\|p^*_{n,N}(k\,\cdot\,)\|_{kN,\infty}} = \frac{\|p^*_{n,N}\|_\infty}{\|p^*_{n,N}\|_{N,\infty}} = c_{n,N}.$$

□

This result allows us to define

$$\gamma_q = \gamma_{\frac{N}{n}} := c_{n,N}$$

for any rational index $q > 2$.

COROLLARY 3. *The function $q \mapsto \gamma_q$, defined on $\mathbb{Q} \cap]2,\infty[$, is strictly monotone decreasing.*

Proof. Given $q_1 = \frac{N_1}{n_1} < q_2 = \frac{N_2}{n_2}$, we may assume that $N_1 = N_2 = N$. This implies $n_1 > n_2$, and thus

$$\gamma_{q_1} = c_{n_1,N} = \sup_{p \in T_{n_1}\setminus\{0\}} \frac{\|p\|_\infty}{\|p\|_{N,\infty}} \geq \sup_{p \in T_{n_2}\setminus\{0\}} \frac{\|p\|_\infty}{\|p\|_{N,\infty}} = c_{n_2,N} = \gamma_{q_2}.$$

Furthermore, we may deduce from Theorem 2.(i) that if p^* is (n_2, N)-extremal, it is not (n_1, N)-extremal, hence $\gamma_{q_1} > \gamma_{q_2}$. □

5. EXAMPLES AND FURTHER NOTES

Theorem 2 allows us to determine $c_{n,N}$ explicitly for special cases.

EXAMPLE 1. Let $n = 1$ and $N \geq 3$.
If N is even, an extremal polynomial is given by

$$p_{1,N}(t) = \cos(t - \tfrac{\pi}{N}) \qquad \text{with} \qquad c_{1,N} = \frac{1}{\cos\frac{\pi}{N}}.$$

If N is odd, an extremal polynomial is given by

$$p_{1,N}(t) = \cos(t - \tfrac{\pi}{N}) + \frac{1 - \cos\frac{\pi}{N}}{2} \qquad \text{with} \qquad c_{1,N} = \frac{3 - \cos\frac{\pi}{N}}{1 + \cos\frac{\pi}{N}}.$$

It should be noted that the case of even $N = 2m$ recovers the reference polynomial used by Ehlich and Zeller in order to prove their estimate

$$c_{n,2m} \leq \frac{1}{\cos\frac{\pi n}{2m}}$$

(see [4, Satz 3]). They also state that this bound is sharp if and only if $n|m$, i.e. $m = nm'$, say, and we recover this result from

$$c_{n,2m} = c_{1,2m'} = \frac{1}{\cos\frac{\pi}{2m'}} = \frac{1}{\cos\frac{\pi n}{2m}}. \qquad \square$$

EXAMPLE 2. Let $n = 2$ and $N \geq 5$.
If N is even, we obtain from Corollary 1

$$p_{2,N}(t) = p_{1,N/2}(2\,t) \qquad \text{with} \qquad c_{2,N} = c_{1,N/2}.$$

If N is odd, an extremal polynomial is given by

$$\begin{aligned} p_{2,N}(t) = &\cos\left(2(t-\tfrac{\pi}{N})\right) + 4\sin^2(\tfrac{\pi}{2N})\cos(t-\tfrac{\pi}{N}) \\ &+ \left(1+(-1)^{\frac{N-1}{2}} 2\sin(\tfrac{\pi}{2N})\right)\sin^2(\tfrac{\pi}{2N}) \end{aligned}$$

with

$$c_{2,N} = \frac{p_{2,N}(\frac{\pi}{N})}{p_{2,N}(\pi+\frac{\pi}{N})} = \frac{1+5\sin^2(\frac{\pi}{2N})+(-1)^{\frac{N-1}{2}}2\sin^3(\frac{\pi}{2N})}{1-3\sin^2(\frac{\pi}{2N})+(-1)^{\frac{N-1}{2}}2\sin^3(\frac{\pi}{2N})}. \qquad \square$$

Further examples may be derived in special cases when elaborating on the properties of extremal polynomials. Of course, in numerical calculations we can also use **Remez-type algorithms** in order to construct an N-extremal alternating set as follows. Given Θ_N as in (1.4), choose $2n+1$ points

$$t_0 < t_1 < t_{k_2} < t_{k_3} < \cdots < t_{k_{2n}}$$

from Θ_N, and find the (unique) polynomial $p \in T_n$ satisfying the interpolation conditions

$$\begin{aligned} p(t_0) &= p(t_1) = 1 \\ p(t_{k_j}) &= (-1)^{j-1}, \quad j = 2, \ldots, 2n. \end{aligned}$$

If this polynomial satisfies $\|p\|_{N,\infty} = 1$, then $p = p^*$ is (n, N)-extremal, and

$$c_{n,N} = p^*(t^*) \qquad \text{with } t^* = \frac{1}{2}(t_0 + t_1) \ .$$

Otherwise modify the points $t_{k_2} < t_{k_3} < \cdots < t_{k_{2n}}$ and iterate. From our experience, it is a good idea to start this process with an almost equidistant subset $t_1{=}\frac{2\pi}{N} < t_{k_2} < t_{k_3} < \cdots < t_{k_{2n}} < 2\pi{+}t_0{=}2\pi$ of Θ_N (see Figure 1 again).

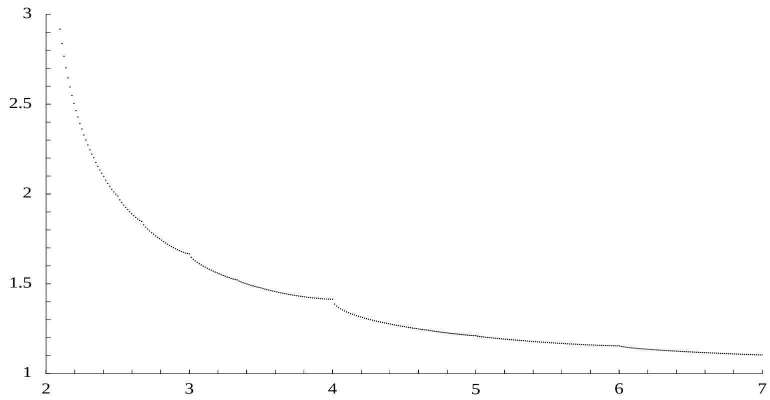

Fig. 2. Values of γ_q (for $q = \frac{N}{72}$, $N = 151 \ldots 504$).

With these methods, it is possible to calculate numerically many values of γ_q efficiently. In doing so, we find surprising results (see Figure 2). At certain numbers like $\frac{8}{3}$, 3, and 4, we observe "cliffs", and also bends at other points like $\frac{5}{2}$ and 6. If we zoom in at these bends (see Figure 3), we realize that they are "cliffs" as well.

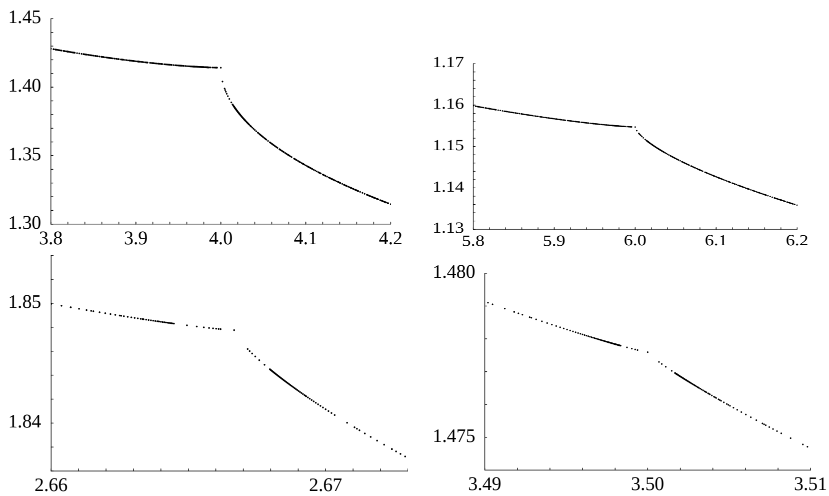

Fig. 3. Values of γ_q around $q = 4$, $q = 6$, $q = \frac{8}{3}$, and $q = \frac{7}{2}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and γ_q is monotone (Corollary 2), we can extend it to a monotone function $]2, \infty[\to]1, \infty[$ by defining

$$\gamma(x) := \inf_{q \le x} \gamma_q.$$

This ensures $\gamma(q) = \gamma_q$ for $q \in \mathbb{Q}$, and makes γ continuous from the left at all irrational points. Figures 2 and 3 lead us to the following conjecture.

CONJECTURE 1. *The function γ has the following properties:*

(i) *γ is continuous.*
(ii) *The left-sided derivative of γ exists at all points, but γ is not differentiable at any rational point.*
(iii) *The graph of γ is self-similar and has fractal dimension.*

It might even be true that at all rational points, the left-sided derivative is 0 and the right-sided derivative is $-\infty$.

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