REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 31 (2002) no. 1, pp. 103–108
ictp.acad.ro/jnaat

ON THE EXTREMAL SEMI-LIPSCHITZ FUNCTIONS

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Abstract. The extremal elements of the unit balls of Banach spaces play an important role in the study of the geometry of the space as well in various applications. For Banach spaces of Lipschitz real functions the extremal elements of the unit ball are investigates in numerous papers (S. Cobzas 1989, J. D. Farmer 1994, N. V. Rao and A. C. Roy 1970, Roy 1968 and in the references therein). In this note we shall present a procedure to obtain extremal elements of the unit ball of the quasi-normed semilinear space of real-valued semi-Lipschitz functions defined on a quasi-metric space.

MSC 2000. 46A22, 26A16.

Keywords. semi metric spaces, semi Lipschitz real functions.

1. INTRODUCTION

Let X be a nonvoid set. A function $d : X \to [0,\infty]$ is called a *quasi - metric* if it satisfies the conditions:

(i) $d(x,y) = d(y,x) = 0 \Longleftrightarrow x = y$
(ii) $d(x,y) \le d(x,z) + d(z,y)$

or

(i′) $d(x,y) = 0 \Longleftrightarrow x = y$

and (ii), for all $x,y,z \in X$. The pair (X,d) is called a *quasi - metric space.*

Remark that d is not a symmetric function, i.e., it is possible that $d(x,y) \neq d(y,x)$ for $x,y \in X$.

A function $f : X \to \mathbb{R}$, defined on a quasi - metric space (X,d) is called *semi-Lipschitz* if there exists $K \ge 0$ such that

$$f(x) - f(y) \le K \cdot d(x,y), \tag{1}$$

for all $x,y \in X$.

A function $f : X \to \mathbb{R}$ is called $\le_d$- *increasing* if

a) $d(x,y) = 0$ implies $f(x) - f(y) \le 0$

or, equivalently

a′) $f(x) - f(y) > 0$ implies $d(x,y) > 0$, for all $x,y \in X$.

Let

$$SLip\,X = \{f : X \to \mathbb{R} \mid f \text{ is } \le_d \text{-increasing and } \|f\|_X < \infty\}, \tag{2}$$

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(see [12]), where

$$\|f\|_X = \sup\left\{\frac{(f(x) - f(y)) \vee 0}{d(x,y)} : x, y \in X, d(x,y) \neq 0\right\}. \tag{3}$$

The set $SLipX$ defined in (2) is exactly the set of all semi-Lipschitz functions on (X, d), and $\|f\|_X$ defined by (3) is the *least semi-Lipschitz* constant for f, i.e.

$$f(x) - f(y) \leq \|f\|_X \cdot d(x,y), \quad x, y \in X, \tag{4}$$

and every $K \geq 0$, for which the inequality (1) holds, satisfies $K \geq \|f\|_X$ (see [9] and [12]).

For $x_0 \in X$ be fixed, denote by

$$SLip_0\, X = \{f \in SLip\, X : f(x_0) = 0\} \tag{5}$$

the set of all real-valued semi-Lipschitz functions defined on the quasi-metric space X which vanish at the fixed point $x_0 \in X$.

Let V be a nonvoid set and $\mathbb{R}^+ = [0, \infty)$. Suppose that on V is defined an operation

$$+ : V \times V \to V$$

such that $(V, +)$ is an Abelian semigroup, i.e. $+$ satisfies the conditions

(i) $(x + y) + z = x + (y + z)$
(ii) $x + y = y + x$
(iii) $0 + x = x$ (0 is the neutral element of semigroup $(V, +)$)

for all $x, y, z \in V$, and an operation

$$\cdot : \mathbb{R}^+ \times V \to V$$

having the properties

(i) $a \cdot (b \cdot x) = (a \cdot b) \cdot x,\ a, b \in \mathbb{R}^+;\ x \in V$
(ii) $(a + b) \cdot x = (a \cdot x) + (b \cdot x),\ a, b \in \mathbb{R}^+;\ x \in V$
(iii) $a \cdot (x + y) = a \cdot x + a \cdot y, a \in \mathbb{R}^+;\ x, y \in V$
(iv) $1 \cdot x = x,\ 1 \in \mathbb{R}^+,\ x \in V.$

The system $(V, +, \cdot, \mathbb{R}^+)$ is called a *semi linear space.*

The opposite element (if exists) of $x \in V$ is denoted by $-x$.

A functional $\|\cdot\|_V : V \to [0, \infty)$ defined on a semilinear space $(V, +, \cdot, \mathbb{R}^+)$ is called a *quasi-norm* on V if it satisfies the conditions:

(i) $x, -x \in V$ and $\|x\|_V = \|-x\|_V = 0 \Longleftrightarrow x = 0$
(ii) $\|ax\|_V = a\|x\|_V,\ a \in \mathbb{R}^+, x \in V$
(iii) $\|x + y\|_V \leq \|x\|_V + \|y\|_V,\ x, y \in V.$

The pair $(V, \|\cdot\|_V)$ is called a *quasi-normed semilinear* space (see [5] and [12]).

If X is a linear space then a functional $\|\cdot\|_X : X \to [0, \infty)$ satisfying the axioms of a quasi-norm is called an *asymmetric norm* on X (see [4]).

It is immediate that the functional defined by (3) is a quasi-norm on $SLip_0\, X$, i.e. the pair $(SLip_0\, X, \|\cdot\|_X)$ is a quasi-normed semilinear space.

If $Y \subset X$ and $x_0 \in Y$ then one considers the semi-Lipschitz functions on Y which vanish at x_0 and the quasi-normed semilinear space $(S\, Lip_0\, Y, \|\cdot\|_Y)$, where $\|\cdot\|_Y$ is defined like in (3) with Y instead of X.

The following extension theorem for semi-Lipschitz functions is similar to Mc Shane's [6] extension theorem for Lipschitz functions.

Theorem 1. [9]. *Let (X, d) be a quasi-metric space, $x_0 \in X$ fixed and $Y \subset X$ such that $x_0 \in Y$. Then every function $f \in SLip_0\, Y$ admits at least one extension in $SLip_0\, X$, i.e. there exists $H \in SLip_0\, X$ such that*

$$H|_Y = f \text{ and } \|H\|_X = \|f\|_Y\,. \tag{6}$$

Denote by

$$E_Y\,(f) = \{H \in SLip_0 X : H|_Y = f \text{ and } \|H\|_X = \|f\|_Y\} \tag{7}$$

the nonvoid set of all extensions of $f \in SLip_0 Y$ which preserve the quasi-norm of f.

We have shown in [9] that the functions

$$F(x) = \inf_{y\in Y} \{f(y) + \|f\|_Y d\,(x, y)\}\,, \quad x \in X, \tag{8}$$

and

$$G\,(x) = \sup_{y\in Y} \{f\,(y) - \|f\|_Y\, d\,(y, x)\}\,, \quad x \in X, \tag{9}$$

belong to $E_Y\,(f)$.

Let

$$B_Y = \{f \in SLip_0\, Y : \|f\|_Y \le 1\} \tag{10}$$

be the unit ball of the quasi-normed semilinear space $(SLip_0\, Y, \|\cdot\|_Y)$ and let B_X be the corresponding unit ball of $(SLip_0\, Y, \|\cdot\|_X)$.

Obviously that $f \in B_Y$ implies $E_Y\,(f) \subset B_X$.

A subset C of a semi-linear space $(V, +, \cdot, \mathbb{R}^+)$ is called *convex* if $\alpha x + (1-\alpha)\, y \in C$ whenever $x, y \in C$ and $\alpha \in [0, 1]$.

A subset M of C is called a *face* of C if $\lambda x + (1-\lambda)\, y \in M$ for $x, y \in C$ and some $\lambda \in (0, 1)$ implies $x, y \in M$. A one-point face of C is called an *extremal element* of C, and the set of all extremal elements of C is denoted by $ext\, C$.

It is obvious that B_Y (respectively B_X) is a convex subset of $SLip_0\, Y$ (respectively $SLip_0\, X$), and if $M \subset B_X$ is a face, then $\|f\|_X = 1$ for any $f \in M$.

Theorem 2. *Let (X, d) be a quasi-metric space, x_0 a fixed point in X, and $Y \subset X$ such that $x_0 \in Y$. Then:*

a) *For every $f \in SLip_0\, Y$ the set $E_Y\,(f) \subset SLip_0\, X$ is convex;*

b) *For every $H \in E_Y\,(f)$ the inequalities*

$$F\,(x) \ge H\,(x) \ge G\,(x)\,, \tag{11}$$

hold for all $x \in X$, where the functions F and G are defined by (8) *and* (9), *respectively;*

c) *If $f \in ext\, B_Y$ then $E_Y(f)$ is a face of B_X and the functions F, G (defined by* (8) *and* (9)*) are extremal elements of B_X.*

Proof. a) Let $F_1, F_2 \in E_Y(f)$ and $\alpha \in (0,1)$. We have

$$(\alpha F_1 + (1-\alpha) F_2)|_Y = \alpha f + (1-\alpha) f = f$$

and

$$\begin{aligned}\|\alpha F_1 + (1-\alpha) F_2\|_X &\le \alpha \|F_1\|_X + (1-\alpha)\|F_2\|_X \\ &= \alpha \|f\|_Y + (1-\alpha)\|f\|_Y = \|f\|_Y .\end{aligned}$$

On the other hand

$$\begin{aligned}\|f\|_Y &= \|\alpha f + (1-\alpha) f\|_Y \\ &= \|\alpha F_1|_Y + (1-\alpha) F_2|_Y\| \le \|\alpha F_1 + (1-\alpha) F_2\|_X\end{aligned}$$

showing that $\|\alpha F_1 + (1-\alpha) F_2\|_X = \|f\|_Y$. It follows $\alpha F_1 + (1-\alpha) F_2 \in E_Y(f)$.

b) Let $H \in E_Y(f)$ and $x \in X$. We have, for any $y \in Y$, $H(x) - f(y) = H(x) - H(y) \le \|H\|_X \cdot d(x,y) = \|f\|_Y \cdot d(x,y)$ so that

$$H(x) \le f(y) + \|f\|_Y\, d(x,y).$$

Taking the infimum with respect to $y \in Y$ we find

$$H(x) \le F(x), \quad \text{for all } x \in X.$$

Also, we have

$$H(y) - H(x) \le \|H\|_X \cdot d(y,x) = \|f\|_Y \cdot d(y,x)$$

which implies

$$H(x) \ge H(y) - \|f\|_Y \cdot d(y,x) = f(y) - \|f\|_Y \cdot d(y,x).$$

Taking the supremum with respect to $y \in Y$ we get

$$H(x) \ge G(x), \quad x \in X.$$

c) Let $f \in ext\ B_Y$. If $F_1, F_2 \in B_X$ and $\lambda \in (0,1)$ are such that $\lambda F_1 + (1-\lambda) F_2 \in E_Y(f)$ then $\lambda F_1|_Y + (1-\lambda) F_2|_Y = f$. Since $f \in extB_Y$ this implies $F_1|_Y = F_2|_Y = f$. Obviously that $\|F_1\|_X = \|F_2\|_X = \|f\|_Y = 1$, showing that $F_1, F_2 \in E_Y(f)$. It follows that $E_Y(f)$ is a face of B_X.

We remark that F, G defined by (8), (9)are extremal elements of $E_Y(f)$.

Indeed, if $H_1, H_2 \in E_Y(f)$ and $\lambda \in (0,1)$ are such that $\lambda H_1 + (1-\lambda) H_2 = F$ then $\lambda H_1|_Y + (1-\lambda) H_2|_Y = f$ and because $f \in extB_Y$ it follows $H_1|_Y = H_2|_Y = f = F|_Y$.

On the other hand $\lambda H_1(x) + (1-\lambda) H_2(x) = F(x), \ \ x \in X$ implies

$$\lambda (H_1(x) - F(x)) + (1-\lambda)(H_2(x) - F(x)) = 0, \quad x \in X$$

and because $H_1(x) \le F(x)$, $H_2(x) \le F(x)$, $x \in X$ and $\lambda \in (0,1)$ it follows

$$H_1(x) = F(x), \quad x \in X,$$
$$H_2(x) = F(x), \quad x \in X.$$

Consequently $F \in ext\, E_Y(f)$. Analogously one obtains $G \in ext E_Y(f)$.

Now let be given $U_1, U_2 \in B_X$ and $\lambda \in (0,1)$ such that $\lambda U_1 + (1-\lambda) U_2 = F$. Then $\lambda U_1|_Y + (1-\lambda) U_2|_Y = F|_Y = f \in ext\, B_Y$ implies $U_1|_Y = U_2|_Y = f$ and $\|U_1|_Y\|_Y = \|U_2|_Y\|_Y = \|f\| = 1$ implies $\|U_1\|_X = \|U_2\|_X = 1$.

It follows that $U_1, U_2 \in E_Y(f)$ and because $F \in ext\, E_Y(f)$ one obtains $U_1 = U_2 = F$. It follows that $F \in ext\, B_X$ and, analogously $G \in ext\, B_X$. □

Remarks. 1°. The reverse implication in c) is also true: if $E_Y(f)$ is a face of B_X then $f \in B_Y$.

Indeed, if $\|f\|_Y = 1$ but $f \notin ext\, B_Y$, then there exist $f_1, f_2 \in B_Y$, $f_1 \ne f_2$, and $\lambda \in (0,1)$ such that $\lambda f_1 + (1-\lambda) f_2 = f$.

Let $F_1' \in E_Y(f_1)$ and $F_2' \in E_Y(f_2)$. Because

$$\lambda F_1'|_Y + (1-\lambda) F_2'|_Y = f$$

and

$$1 = \|\lambda F_1'|_Y + (1-\lambda) F_2'|_Y\| \le \|\lambda F_1' + (1-\lambda) F_2'\|_X \le 1$$

we have

$$\|\lambda F_1' + (1-\lambda) F_2'\|_X = 1,$$

showing that $\lambda F_1' + (1-\lambda) F_2' \in E_Y(f)$. Since $F_1'|_Y = f_1 \ne f_2 = F_2'|_Y$ it follows that $E_Y(f)$ is not a face of B_X.

2°. The assertion c) from Theorem 2 gives us a way to obtain extremal elements of B_X, namely as the extensions (8) and (9) of extremal elements of B_Y. □

Example. Consider the quasi metric space $(\mathbb{R}, d)$, where $\mathbb{R}$ is the set of real numbers and

$$d(x,y) = \begin{cases} x - y & \text{if } x \ge y \\ 1 & \text{if } x < y. \end{cases}$$

For $Y = \{0,1,2\}$ and $x_0 = 0$ consider the semilinear spaces $SLip_0\, Y$ and $SLip_0\, X$ equipped with the quasi-norms of the type (3).

The function $f(y) = y$, $y \in \{0,1,2\} = Y$ is an extremal element of B_Y. Observe that for any $h \in B_Y$ we have $h(1) \le f(1) = 1$ and $h(2) \le f(2) = 2$, because, if contrary, i.e. $h(1) > f(1)$ or $h(2) > f(2)$, then $\|h\|_Y > 1$. If $f_1, f_2 \in B_Y$ and $\alpha \in (0,1)$ are such that $\alpha f_1 + (1-\alpha) f_2 = f$ then, taking into account the relations $f_i(0) = f(0) = 0$, $f_i(1) \le f(1)$, $f_i(2) \le f(2)$, $i = 1,2$, we get $\alpha(f_1 - f) + (1-\alpha)(f_2 - f) = 0$ implying $f_1 = f_2 = f$.

In this case the extensions F and G, given by (8) and (9), are

$$F(x) = \begin{cases} 1, & \text{for } x \in (-\infty, 0) \\ x, & \text{for } x \in [0, +\infty) \end{cases} \text{ respectively } G(x) = \begin{cases} x, & \text{for } x \in (-\infty, 2] \\ 1, & \text{for } x \in (2, +\infty) \end{cases}$$

and they are extremal elements of B_X. □

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Received by the editors: September 26, 2001.