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AITKEN-STEFFENSEN TYPE METHODS FOR NONDIFFERENTIABLE FUNCTIONS (I)*

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Abstract. We study the convergence of the Aitken-Steffensen method for solving a scalar equation $f(x) = 0$. Under reasonable conditions (without assuming the differentiability of f) we construct some auxilliary functions used in these iterations, which generate bilateral sequences approximating the solution of the considered equation.

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1. INTRODUCTION

In this note we shall deal with the construction of the auxiliary functions appearing in the Aitken-Steffensen-type methods for solving the equation:

$$f(x) = 0, \tag{1}$$

where $f : [a,b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$. Since we shall not assume differentiability conditions on f, we shall consider instead the first and second order divided differences of f, denoted by $[u,v;f]$, resp. $[u,v,w;f]$, $u,v,w \in [a,b]$.

Let $g, g_i : [a,b] \to \mathbb{R}$, $i = 1,2$, be three functions such that the equations

$$x - g(x) = 0 \quad \text{and} \tag{2}$$

$$x - g_i(x) = 0, \; i = 1,2 \tag{3}$$

are equivalent to (1).

The following three Aitken-Steffensen methods are well known:

1. The Steffensen method, which generates two sequences, $(x_n)_{n\geq 1}$ and $(g(x_n))_{n\geq 1}$, by

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, \quad n = 1,2,\ldots, \; x_1 \in [a,b], \tag{4}$$

where g is given by (2).

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2. The Aitken method, which generates the sequences $(x_n)_{n\geq 1}$, $(g_i(x_n))_{n\geq 1}$ $i = 1, 2$, by

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(x_n); f]}, \quad n = 1, 2, \ldots, \ x_n \in [a, b], \tag{5}$$

with g_1, g_2 given by (3).

3. The Aitken-Steffensen method, which generates the sequences $(x_n)_{n\geq 1}$, $(g_1(x_n))_{n\geq 1}$, and $(g_2(g_1(x_n)))_{n\geq 1}$, by

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(g_1(x_n)); f]}, \quad n = 1, 2, \ldots, \ x_1 \in [a, b]. \tag{6}$$

A certain method presents an important advantage particularly when it yields sequences approximating the solution both from the left and from the right. In such a case we obtain a rigorous control if the error at each iteration step.

We shall study the choice of the functions g, g_1, g_2 such that the above methods to yield bilateral approximations for the solution of (1).

Regarding the monotonicity and the convexity of the function f we shall use the following definitions. The function f is nondecreasing (increasing) on $[a, b]$ if $[u, v; f] \geq 0$ (> 0) for all $u, v \in [a, b]$. The function f is nonconcave (convex) on $[a, b]$ if $[u, v, w; f] \geq 0$ (> 0) for all $u, v, w \in [a, b]$.

Let $x_0 \in [a, b]$ and $p_{x_0} : [a, b] \setminus \{x_0\} \to \mathbb{R}$, given by

$$p_{x_0}(x) = [x_0, x; f]. \tag{7}$$

The following result was proved in [3, p. 290].

THEOREM 1. a) *If the function f is nonconcave on $[a, b]$, the p_{x_0} is a nondecreasing function on $[a, b] \setminus \{x_0\}$.*

b) *If f is a convex function on $[a, b]$, then p_{x_0} is an increasing function on $[a, b] \setminus \{x_0\}$.*

In other words, if f is nonconcave (convex) on $[a, b]$, then for all $x', x'' \in [a, b] \setminus \{x_0\}$, $x' < x''$ one obtains

$$[x_0, x'; f] \leq (<) \, [x_0, x''; f]. \tag{8}$$

Consider now $u, v, w, t \in [a, b]$ such that $u = \min\{u, v, w, t\}$ and $t = \max\{u, v, w, t\}$. The following lemma holds.

LEMMA 2. *If f is nonconcave (convex) on $[a, b]$, then for all $v, w \in (u, t)$, $v \neq w$, one has*

$$[u, v; f] \leq (<) \, [w, t; f]. \tag{9}$$

Proof. We shall consider only the case " $\leq$ ", since the other one is similarly obtained. There an two alternatives:

Case I. $u < v < w < t$. Taking into account the symmetry of the divided differences with respect to the nodes and Theorem 1 we get $[u, v; f] \leq [u, w; f] = [w, u; f] \leq [w, t; f]$.

Case II. $u < w < v < t$. As above, we obtain: $[u,v;f] \leq [u,t;f] = [t,u;f] \leq [t,w;f] = [w,t;f]$. □

2. THE CONVERGENCE OF THE AITKEN-STEFFENSEN-LIKE METHOD

We shall make the following assumptions on f :

i. $f(a) \cdot f(b) < 0$;
ii. f is increasing on $[a,b]$;
iii. f is convex on $[a,b]$ and continuous in a and b;
iv. f is differentiable at $\bar{x} \in (a,b)$, where $\bar{x}$ is the solution of (1).

Remark 1. Hypothesis iii. ensures the continuity of f on $[a,b]$ (see [3, p. 295]). □

Remark 2. Hypothesis i.–iii. ensure the existences and the uniqueness of the solution $\bar{x} \in (a,b)$ of the equation (1). □

Let α and β be two numbers such that $a < \alpha < \beta < b$, $f(\alpha) < 0$ and $f(\beta) > 0$. Consider the functions g_1, g_2 given by

$$g_1(x) = x - \frac{f(x)}{[\beta,b;f]}, \quad x \in [\alpha,\beta] \text{ and} \tag{10}$$

$$g_2(x) = x - \frac{f(x)}{[a,\alpha;f]}, \quad x \in [\alpha,\beta]. \tag{11}$$

From hypotheses ii. iii. and applying Lemma 2 it follows that for all $u,v \in (\alpha,\beta)$

$$[u,v;g_1] > 0 \text{ and } [u,v;g_2] < 0. \tag{12}$$

Consider now an initial approximation $x_1 \in (\alpha,\beta)$ satisfying

a) $f(x_1) < 0$;
b) $g_2(g_1(x_1)) < \beta$.

The following result holds regarding the convergence of the sequence (6).

Theorem 3. *If the function f obeys* i.–iv., *the functions g_1 and g_2 are given by* (10) *resp.* (11) *and x_1 satisfies the assumptions* a) *and* b)*, then the sequences $(x_n)_{n\geq1}$, $(g_1(x_n))_{n\geq1}$ and $(g_2(g_1(x_n)))_{n\geq1}$ generated by* (6) *satisfy:*

j. $(x_n)_{n\geq1}$ *is increasing;*
jj. $(g_1(x_n))_{n\geq1}$ *is increasing;*
jjj. $(g_2(g_1(x_n)))_{n\geq1}$ *is decreasing;*
jv. *for all $n \in \mathbb{N}$, $n \geq 1$, the following relations hold:*

$$x_n < g_1(x_n) < x_{n+1} < \bar{x} < g_2(g_1(x_n)). \tag{13}$$

Proof. By ii. and a) it follows that $x_1 < \bar{x}$, and from $f(x_1) < 0$ and $[\beta,b;f] > 0$ we get $g_1(x_1) > x_1$. Since $x_1 < \bar{x}$ and g_1 is increasing, one obtains

$g_1(x) < g_1(\bar{x}) = \bar{x}$, i.e., $x_1 < g_1(x_1) < \bar{x}$. On the other hand, by b) and (6) it follows

$$x_2 = g_1(x_1) - \frac{f(g_1(x_1))}{[g_1(x_1), g_2(g_1(x_1)); f]}. \tag{14}$$

Since $g_1(x_1) < \bar{x}$ we get $f(g_1(x_1)) < 0$ and hence $x_2 > g(x_1)$. The fact that g_2 is decreasing and $g_1(x_1) < \bar{x}$ imply that $g_2(g_1(x_1)) > g_2(\bar{x}) = \bar{x}$. It follows that $f(g_2(g_1(x_1))) > 0$, and taking into account the equality

$$g_1(x_1) - \tfrac{f(g_1(x_1))}{[g_1(x_1), g_2(g_1(x_1)); f]} = g_2(g_1(x_1)) - \tfrac{f(g_2(g_1(x_1)))}{[g_1(x_1), g_2(g_1(x_1)); f]} = x_2$$

it follows that $x_2 < g_2(g_1(x_1))$ and hence the following identity is true

$$\begin{aligned} f(x_2) =& f(g_1(x_1)) + [g_1(x_1), g_2(g_1(x_1)); f](x_2 - g_1(x_1)) \\ &+ [x_2, g_1(x_1), g_2(g_1(x_1)); f](x_2 - g_1(x_1))(x_2 - g_2(g_1(x_1))), \end{aligned}$$

whence, taking into account the following facts: f is convex, $x_2 > g_1(x_1)$, $x_2 < g_2(g_1(x_1))$ and (14), it follows that $f(x_2) < 0$, i.e., $x_2 < \bar{x}$.

The inequality $x_1 < x_2$ and the fact that g_1 is increasing imply that $g_1(x_1) < g_1(x_2)$. Since g_2 is decreasing we get $g_2(g_1(x_1)) > g_2(g_1(x_2))$. From $x_2 < \bar{x}$ it follows that $g_1(x_2) < \bar{x}$ and $g_2(g_1(x_2)) > g_2(\bar{x}) = \bar{x}$.

Obviously, the above reason may be applied for any x_n, $n \geq 2$, so that the induction principle completes the proof. □

The sequences $(x_n)_{n\geq 1}$, $(g_1(x_n))_{n\geq 1}$ and $(g_2(g_1(x_n)))_{n\geq 1}$ are monotone and bounded, and therefore they converge.

Let $x^* = \lim x_n$, hence $x^* = \sup_{n\in\mathbb{N}}\{x_n\}$, and let $b = \sup_{n\in\mathbb{N}}\{g_1(x_n)\}$. We shall prove that $x^* = b$. The relations $x^* < b$ and $x^* > b$ cannot hold, since, as implied by (13), we get

$$x_n < g_1(x_n) < x_{n+1} < g_1(x_{n+1}), \quad n = 1, 2, \ldots$$

which lead to conclusions contradicting the definition of the exact upper bound. Therefore, the following relations are true: $x^* = \lim g_1(x_n) = g_1(x^*)$, whence, taking into account the equivalence of (1) and (3), it follows that $x^* = \bar{x}$. The equality $\bar{x} = g_2(\bar{x})$ implies $\lim g_2(g_1(x_n)) = g_2(g_1(\bar{x})) = g_2(\bar{x}) = \bar{x}$.

The three sequences have the same limit $\bar{x}$,which is the solution of (1).

By (13) we obtain

$$\bar{x} - x_{n+1} \leq g_2(g_1(x_n)) - x_{n+1},$$

and

$$\bar{x} - x_{n+1} \leq g_2(g_1(x_n)) - g_1(x_n), \; n \in \mathbb{N}^*,$$

which provide a control of the error at each iteration step.

In a forthcoming work we shall present some results regarding the convergence of the Steffensen and Aitken methods.

We end with some remarks.

Remark 3. Since f is convex, then in (10), resp. (11) we may replace the divided differences $[a, \alpha; f]$ and $[\beta, b; f]$ by $f'_r(a)$, resp. $f'_l(b)$. □

Remark 4. The following relations hold:

$$|\bar{x}-x_{n+1}| \leq \frac{l^3 m\,[\beta,b;f]}{[a,\alpha;f]^2}\,|\bar{x}-x_n|^2, \quad n=1,2,\ldots,$$

where $l = 1-\frac{[a,\alpha;f]}{[\beta,b;f]}$ and $m=\sup\{\,[u,v,w;f] : u,v,w\in[\alpha,\beta]\,\}$.

Proof. Consider the following identities:

$$\begin{aligned} f(\bar{x}) =& f(g_1(x_n)) + [g_1(x_n), g_2(g_1(x_n)); f](\bar{x}-g_1(x_n)) \\ &+ [\bar{x}, g_1(x_n), g_2(g_1(x_n)); f](\bar{x}-g_1(x_n))(\bar{x}-g_2(g_1(x_n))), \end{aligned}$$

whence, by (6) and $f(\bar{x})=0$, we get

$$\bar{x}-x_{n+1} = -\frac{[\bar{x}, g_1(x_n), g_2(g_1(x_n)); f]}{[g_1(x_n), g_2(g_1(x_n)); f]}(\bar{x}-g_1(x_n))(\bar{x}-g_2(g_1(x_n))). \tag{15}$$

Next, g_1 and g_2 obey the following identities:

$$\bar{x}-g_1(x_n) = g_1(\bar{x})-g_1(x_n) = [x_n,\bar{x};g_1](\bar{x}-x_n), \tag{16}$$

$$\bar{x}-g_2(g_1(x_n)) = [g_1(x_n), g_1(\bar{x}); g_2]\,[x_n,\bar{x};g_1](\bar{x}-x_n). \tag{17}$$

From the definitions of g_1 and g_2 we deduce

$$0<[x_n,\bar{x};g_1] = 1-\frac{[x_n,\bar{x};f]}{[\beta,b;f]} < 1-\frac{[\alpha,a;f]}{[\beta,b;f]} = l < 1 \tag{18}$$

and

$$\begin{aligned} [g_1(x_n), g_1(\bar{x}); g_2] &= 1-\tfrac{[g_1(x_n),\bar{x};f]}{[\alpha,a;f]} \\ &> 1-\tfrac{[\beta,b;f]}{[\alpha,a;f]} = \tfrac{[\beta,b;f]}{[\alpha,a;f]}\left(\tfrac{[\alpha,a;f]}{[\beta,b;f]}-1\right) = -l\tfrac{[\beta,b;f]}{[\alpha,a;f]}, \end{aligned}$$

whence $-[g_1(x_n), g_1(\bar{x}); g_2] < l\frac{[\beta,b;f]}{[\alpha,a;f]}$.

Since g_2 is nondecreasing we get

$$|\,[g_1(x_n), g_1(\bar{x}); g_2]\,| < l\frac{[\beta,b;f]}{[\alpha,a;f]}. \tag{19}$$

According to Lemma 2

$$[g_1(x_n), g_2(g_1(x_n)); f] > [\alpha,a;f], \quad n=1,2,\ldots$$

and by (15)–(19) we finally get

$$|\bar{x}-x_{n+1}| \leq \frac{ml^3\,[\beta,b;f]}{[a,\alpha;f]^2}\,|\bar{x}-x_n|^2, \quad n=1,2,\ldots$$

□

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