USE OF AN IDENTITY OF A. HURWITZ FOR CONSTRUCTION OF
A LINEAR POSITIVE OPERATOR OF APPROXIMATION

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Abstract. By using a general algebraic identity of Adolf Hurwitz [1], which generalizes an important identity of Abel, we construct a new operator \( S(\beta_1, \ldots, \beta_m) \) approximating the functions. A special case of this is the operator \( Q^m \) of Cheney-Sharma.

We show that this new operator, applied to a function \( f \in C[0,1] \), is interpolatory at both sides of the interval \([0,1]\), and reproduces the linear functions.

We also give an integral representation of the remainder of the approximation formula of the function \( f \) by means of this operator. By applying a criterion of T. Popoviciu [2], is also given an expression of this remainder by means of divided difference of second order.

MSC 2000. 41A10, 41A36.

Keywords. Hurwitz’s identity, Abel’s generalization of the binomial formula, linear positive operator of approximation, the Peano theorem, divided difference.

1. INTRODUCTION

In 1902 A. Hurwitz [1] has given the following identity

\[
(u + v)(u + v + \beta_1 + \ldots + \beta_m)^{m-1} = \sum u(u + \beta_{i_1} + \ldots + \beta_{i_k})^{k-1} v(v + \beta_{j_1} + \ldots + \beta_{j_{m-k}})^{m-k-1}.
\]

In the special case \( \beta_1 = \beta_2 = \ldots = \beta_m = \beta \) it reduces to an identity of Abel-Jensen

\[
(u + v)(u + v + m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u(u + k\beta)^{k-1} v(v + (m - k)\beta)^{m-k-1}.
\]

If we replace in (1) \( u = x \) and \( v = 1 - x \), we obtain the equality

\[
\sum x(x + \beta_{i_1} + \ldots + \beta_{i_k})^{k-1}(1 - x)(1 - x + \beta_{j_1} + \ldots + \beta_{j_{m-k}})^{m-k-1} = (1 + \beta_1 + \ldots + \beta_m)^{m-1}.
\]

By using this relation we can construct a linear positive operator, depending on \( m \) nonnegative parameters, defined for any function \( f \in C[0,1] \), by the

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following formula

\[(2) \ (1 + \beta_1 + \ldots + \beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x) = \sum_{k=0}^{m} w_m^{(\beta_1,\ldots,\beta_m)}(x) f\left(\frac{k}{m}\right),\]

where

\[(3) \ w_m^{(\beta_1,\ldots,\beta_m)}(x) = \sum x(x + \beta_1 + \ldots + \beta_k)^{k-1}(1 - x)(1 - x + \beta_1 + \ldots + \beta_{m-k})^{m-k-1}.\]

It is easy to see that we can write

\[(4) \ (1 + \beta_1 + \ldots + \beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x) = (1 - x)(1 - x + \beta_1 + \ldots + \beta_m)^{m-1}f(0) + x(1 - x) \sum_{k=1}^{m-1} w_m^{(\beta_1,\ldots,\beta_m)}(x) f\left(\frac{k}{m}\right) + x(x + \beta_1 + \ldots + \beta_m)^{m-1}f(1).\]

Now we can observe that the polynomial \((S_m^{(\beta_1,\ldots,\beta_m)}f)(x)\) is interpolatory at both sides of the interval \([0,1]\), for any nonnegative values of the parameters \(\beta_1,\ldots,\beta_m\).

Hence

\[\text{Hence (4) } (S_m^{(\beta_1,\ldots,\beta_m)}f)(0) = f(0), \ (S_m^{(\beta_1,\ldots,\beta_m)}f)(1) = f(1).\]

It follows that our operator reproduces the linear functions.

Consequently, the approximation formula

\[f(x) = (S_m^{(\beta_1,\ldots,\beta_m)}f)(x) + (R_m^{(\beta_1,\ldots,\beta_m)}f)(x)\]

has the degree of exactness one.

By using a known theorem of Peano we can give an integral representation for the remainder of the formula (5).

\[\text{Theorem 1. If } f \in C^2[0,1] \text{ then the remainder term of the approximation formula (5) can be represented under the following integral form}\]

\[(6) \ (R_m^{(\beta_1,\ldots,\beta_m)}f)(x) = \int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t;x) f''(t)dt,\]

where the Peano kernel is

\[G_m^{(\beta_1,\ldots,\beta_m)}(t;x) = (R_m^{(\beta_1,\ldots,\beta_m)}\varphi_x)(t),\]

with

\[\varphi_x(t) = \frac{x - t + |x - t|}{2} = (x - t)_+.\]

\[\text{Proof. We have}\]

\[G_m^{(\beta_1,\ldots,\beta_m)}(t;x) = (x - t)_+ - \sum_{k=0}^{m} w_m^{(\beta_1,\ldots,\beta_m)}(x) \left(\frac{k}{m} - t\right)_+.\]
By writing the explicit expression for the Peano kernel we can observe that this kernel represents a polygonal line situated beneath the $t$-axis, which joins the points $(0,0)$ and $(0,1)$.

If we apply the first law of the mean to the integral, we get

$$(R_m^{(\beta_1,\ldots,\beta_m)} f)(x) = f''(\xi) \int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t; x) dt$$

and the formula (5) becomes

$$(7) \quad f(x) = (S_m^{(\beta_1,\ldots,\beta_m)} f)(x) + f''(\xi) \int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t; x) dt.$$

If we now replace here $f(x) = e_2(x) = x^2$, we find

$$x^2 = (S_m^{(\beta_1,\ldots,\beta_m)} e_2)(x) = 2 \int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t; x) dt.$$

Consequently, we can write

$$\int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t; x) dt = \frac{1}{2} \left[ x^2 - (S_m^{(\beta_1,\ldots,\beta_m)} e_2)(x) \right] = \frac{1}{2} (R_m^{(\beta_1,\ldots,\beta_m)} e_2)(x). \quad \square$$

**Theorem 2.** If $f \in C^2[0,1]$ then we can give the following expression for the remainder of formula (5):

$$(8) \quad (R_m^{(\beta_1,\ldots,\beta_m)} f)(x) = \frac{1}{2} (R_m^{(\beta_1,\ldots,\beta_m)} e_2)(x) f''(\xi),$$

where $\xi \in (0,1)$.

**Proof.** Since $R_m^{(\beta_1,\ldots,\beta_m)} f = 0$ if $f$ is a linear function and it is different from zero for any convex function of the first order, we can apply a criterion of T. Popoviciu [2] and we can conclude that this remainder is of simple form and we can state the following result. \quad \square

**Theorem 3.** If the second-order divided differences of the function $f$ are bounded on the interval $[0,1]$, then there exist three distinct points $t_{m,1}, t_{m,2}, t_{m,3}$ in $[0,1]$, such that the remainder can be represented under the form

$$(9) \quad (R_m^{(\beta_1,\ldots,\beta_m)} f)(x) = (R_m^{(\beta_1,\ldots,\beta_m)} e_2)(x)[t_{m,1}, t_{m,2}, t_{m,3}; f]$$

**Proof.** It is clear that if $f \in C^2[0,1]$ we can apply the mean-value theorem of divided differences and we can obtain formula (8). \quad \square

**REFERENCES**


Received by the editors: February 10, 2002.