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USE OF AN IDENTITY OF A. HURWITZ FOR CONSTRUCTION OF A LINEAR POSITIVE OPERATOR OF APPROXIMATION

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Abstract. By using a general algebraic identity of Adolf Hurwitz [1], which generalizes an important identity of Abel, we construct a new operator $S_m^{(\beta_1,\ldots,\beta_m)}$ approximating the functions. A special case of this is the operator Q_m^β of Cheney-Sharma.

We show that this new operator, applied to a function $f \in C[0,1]$, is interpolatory at both sides of the interval $[0,1]$, and reproduces the linear functions.

We also give an integral representation of the remainder of the approximation formula of the function f by means of this operator. By applying a criterion of T. Popoviciu [2], is also given an expression of this remainder by means of divided difference of second order.

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1. INTRODUCTION

In 1902 A. Hurwitz [1] has given the following identity

$$(1)\qquad (u+v)(u+v+\beta_1+\ldots+\beta_m)^{m-1} = \\ = \sum u(u+\beta_{i_1}+\ldots+\beta_{i_k})^{k-1} v(v+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.$$

In the special case $\beta_1=\beta_2=\ldots=\beta_m=\beta$ it reduces to an identity of Abel-Jensen

$$(u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1} v(v+(m-k)\beta)^{m-k-1}.$$

If we replace in (1) $u=x$ and $v=1-x$, we obtain the equality

$$\sum x(x+\beta_{i_1}+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1} = \\ = (1+\beta_1+\ldots+\beta_m)^{m-1}.$$

By using this relation we can construct a linear positive operator, depending on m nonnegative parameters, defined for any function $f \in C[0,1]$, by the

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following formula

$$(2)\qquad (1+\beta_1+\ldots+\beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)=\sum_{k=0}^{m} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)f(\tfrac{k}{m}),$$

where

$$(3)\quad w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)= \\ =\sum x(x+\beta_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.$$

It is easy to see that we can write

$$(1+\beta_1+\ldots+\beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)= \\ =(1-x)(1-x+\beta_1+\ldots+\beta_m)^{m-1}f(0)+ \\ +x(1-x)\sum_{k=1}^{m-1} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)f(\tfrac{k}{m})+x(x+\beta_1+\ldots+\beta_m)^{m-1}f(1).$$

Now we can observe that the polynomial $(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)$ is interpolatory at both sides of the interval $[0,1]$, for any nonnegative values of the parameters $\beta_1,\ldots,\beta_m$.

Hence

$$(4)\qquad (S_m^{(\beta_1,\ldots,\beta_m)}f)(0)=f(0),\ (S_m^{(\beta_1,\ldots,\beta_m)}f)(1)=f(1).$$

It follows that our operator reproduces the linear functions.

Consequently, the approximation formula

$$(5)\qquad f(x)=(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)+(R_m^{(\beta_1,\ldots,\beta_m)}f)(x)$$

has the degree of exactness one.

By using a known theorem of Peano we can give an integral representation for the remainder of the formula (5).

Theorem 1. *If $f\in C^2[0,1]$ then the remainder term of the approximation formula* (5) *can be represented under the following integral form*

$$(6)\qquad (R_m^{(\beta_1,\ldots,\beta_m)}f)(x)=\int_0^1 G_m^{(\beta_1,\ldots,\beta_m)}(t;x)f''(t)\mathrm{d}t,$$

where the Peano kernel is

$$G_m^{(\beta_1,\ldots,\beta_m)}(t;x)=(R_m^{(\beta_1,\ldots,\beta_m)}\varphi_x)(t),$$

with

$$\varphi_x(t)=\frac{x-t+|x-t|}{2}=(x-t)_+.$$

Proof. We have

$$G_m^{(\beta_1,\ldots,\beta_m)}(t;x)=(x-t)_+-\sum_{k=0}^{m} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)\left(\tfrac{k}{m}-t\right)_+.$$

By writing the explicit expression for the Peano kernel we can observe that this kernel represents a polygonal line situated beneath the t-axis, which joins the points $(0,0)$ and $(0,1)$.

If we apply the first law of the mean to the integral, we get

$$(R_m^{(\beta_1,\dots,\beta_m)}f)(x) = f''(\xi)\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t;x)\mathrm{d}t$$

and the formula (5) becomes

$$f(x) = (S_m^{(\beta_1,\dots,\beta_m)}f)(x) + f''(\xi)\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t;x)\mathrm{d}t. \tag{7}$$

If we now replace here $f(x) = e_2(x) = x^2$, we find

$$x^2 = (S_m^{(\beta_1,\dots,\beta_m)}e_2)(x) = 2\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t;x)\mathrm{d}t.$$

Consequently, we can write

$$\int_0^1 G_m^{(\beta_1,\dots,\beta_m)}(t;x)\mathrm{d}t = \tfrac{1}{2}\left[x^2 - (S_m^{(\beta_1,\dots,\beta_m)}e_2)(x)\right] = \tfrac{1}{2}(R_m^{(\beta_1,\dots,\beta_m)}e_2)(x). \quad \square$$

THEOREM 2. *If $f \in C^2[0,1]$ then we can give the following expression for the remainder of formula* (5):

$$(R_m^{(\beta_1,\dots,\beta_m)}f)(x) = \tfrac{1}{2}(R_m^{(\beta_1,\dots,\beta_m)}e_2)(x)f''(\xi), \tag{8}$$

where $\xi \in (0,1)$.

Proof. Since $R_m^{(\beta_1,\dots,\beta_m)}f = 0$ if f is a linear function and it is different from zero for any convex function of the first order, we can apply a criterion of T. Popoviciu [2] and we can conclude that this remainder is of simple form and we can state the following result. $\square$

THEOREM 3. *If the second-order divided differences of the function f are bounded on the interval $[0,1]$, then there exist three distinct points $t_{m,1},\ t_{m,2}$, $t_{m,3}$ in $[0,1]$, such that the remainder can be represented under the form*

$$(R_m^{(\beta_1,\dots,\beta_m)}f)(x) = (R_m^{(\beta_1,\dots,\beta_m)}e_2)(x)[t_{m,1},\ t_{m,2},\ t_{m,3}; f] \tag{9}$$

Proof. It is clear that if $f \in C^2[0,1]$ we can apply the mean-value theorem of divided differences and we can obtain formula (8). $\square$

REFERENCES

[1] HURWITZ, A., *Über Abel's Vereingemeinerung der Binomischen Formel*, Acta Mathematica, **26**, pp. 199–203, 1902.

[2] POPOVICIU, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica (Cluj), **1** (**24**), pp. 95–142, 1959.

[3] STANCU, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17**, pp. 270–278, 1963.

[4] STANCU, D. D. and CISMAŞIU, C., *On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numér. Théor. Approx., **26**, pp. 221–227, 1997.

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